THE SINGLE CONSERVATION LAW: ASYMPTOTICS

In this lecture we study the behavior of entropy solutions at $t\nearrow\infty$. Intuitively, the total variation of the solution for each fixed t decreases as $t\nearrow\infty$. Therefore we expect converging behavior as $t\nearrow\infty$.

1. Uniform decay for initial values with compact support.

First note that we can always replace f by $f-c$ and also do the change of variables $x'=x-f'(0)\,t$ such that the equation reduces to

$$u_t+f(u)_x=0, \qquad u\,|_{t=0}=u_0 \tag{1}$$

with $f(0)=f'(0)=0$. Recall that we always assume $f''>0$.

We will prove the following theorem.

Theorem 1. (Uniform decay) *Assume that $f''>0$, $f(0)=f'(0)=0$, and that $u_0(x)$ is a bounded measurable function having compact support. Then the unique entropy solution decays to 0 uniformly at a rate $t^{-1/2}$ as $t\nearrow\infty$.*

- Main idea.

  Since $f''>0$, there is $\mu>0$ such that $f''>\mu$ for all $u\in\left[-\|u_0\|_{L^\infty},\|u_0\|_{L^\infty}\right]$ which leads to

  $$f'(u)=f'(u)-f'(0)=f''(\xi)\,u \qquad\Longrightarrow\qquad |u|\leqslant|f'(u)|/\mu. \tag{2}$$

  Now recall the equation for (backward) characteristics,

  $$x=x_0+f'(u_0(x_0))\,t \quad\Longrightarrow\quad |f'(u_0(x_0))|\leqslant\frac{|x-x_0|}{t}. \tag{3}$$

  Combine the above, we obtain

  $$|u|\leqslant c\,\frac{|x-x_0|}{t}. \tag{4}$$

  The desired result follows if we can show $|x-x_0|<C\,t^{1/2}$ for all (x,t) in the support of u. It suffices to show that the support of $u(\cdot,t)$ grows no faster than $C t^{1/2}$.

- A technical remark.

  The equation for characteristics only holds in the smooth part of the solution. According to a theorem by R. DiPerna,[1] when $f''>0$ the solution is always piecewise smooth. Furthermore note that for any entropy solution, the backward characteristic can always reach $t=0$.

- Now we try to obtain the bound on the growth of the support of $u(\,\cdot\,,\,t)$. Denote by $s_+(t)$ the infimum of x such that $u(y,t)=0$ for all $y>x$. $s_-(t)$ is defined similarly. Thus the goal is to show

  $$s_+(t)-s_-(t)\leqslant C\,t^{1/2}. \tag{5}$$

  Fix $t=T$. Note that $u(s_+(T)+,T)=0$. Now if $u(s_+(T)-,T)=0$, then we have $\frac{\mathrm{d}}{\mathrm{d}t}s^+(t)=0$ at $t=T$;

  If $u(s_+(T)-,T)>0$ (<0 is prohibited by the entropy condition), consider the region enclosed by the backward characteristic from $(s_+(T),T)$, $t=0$, and $x=s_+(T)$. Call the three curves $\Gamma_1,\Gamma_2,\Gamma_3$.

  Integrating the equation

  $$u_t+f(u)_x=0 \tag{6}$$

  over this region and using the jump condition, we obtain

  $$\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3} u\,\mathrm{d}x-f(u)\,\mathrm{d}t=0. \tag{7}$$

1. R. J. DiPerna, Singularities of solutions of nonlinear hyperbolic systems of conservation laws, Arch. Rat. Mech. Anal., 60, 75–100, 1975.

Now along Γ_1, $u = u(s_+(T)+, T)$ (denote by $\bar{u}$) which gives

$$\int_{\Gamma_1} = \int_0^T [u\, f'(u) - f]\, \mathrm{d}t = T\, [\bar{u}\, f'(\bar{u}) - f(\bar{u})]. \tag{8}$$

Along Γ_2 we have

$$\int_{\Gamma_2} = -\int_y^\infty u_0(x)\, \mathrm{d}x \tag{9}$$

where y is the intersection of Γ_1 and $t = 0$.

Along Γ_3 we have

$$\int_{\Gamma_3} = 0. \tag{10}$$

Therefore we have

$$T\, [\bar{u}\, f'(\bar{u}) - f(\bar{u})] \leqslant \max_y \int_y^\infty u_0(x)\, \mathrm{d}x. \tag{11}$$

Expanding f at 0, we have

$$u(s_+(T)+, T) \leqslant C\, t^{-1/2} \tag{12}$$

which leads to

$$s^+ \leqslant C\, t^{1/2}. \tag{13}$$

The bound on s^- is estimated similarly.

2. Asymptotic profile for initial values with compact support.

We use the following notation:

$$q \equiv \max_y \int_y^\infty u_0(x)\, \mathrm{d}x, \quad -p \equiv \min_y \int_{-\infty}^y u_0(x)\, \mathrm{d}x, \quad k = f''(0). \tag{14}$$

Define

$$w(x,t) = \begin{cases} \dfrac{x}{k\, t} & s_- - \sqrt{2\, k\, p}\, t^{1/2} < x < s_+ + \sqrt{2\, k\, q}\, t^{1/2} \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

This function is called an "N-wave" due to its shape at every fixed t. Then we have

$$\|u(\cdot, t) - w(\cdot, t)\|_{L^1(\mathbb{R})} = O\Big(t^{-1/2}\Big) \qquad \text{as} \qquad t \nearrow \infty. \tag{16}$$

In other words, as $t \nearrow \infty$, all the details in the initial data are lost.

To see why such convergence makes sense, we take any (x, t) and let $y(x, t)$ be the intersection of the backward characteristic and the x-axis. Thus we have

$$u(x,t) = u_0(y, t) \tag{17}$$

and

$$x = y(x,t) + f'(u_0(y(x,t)))\, t = y(x,t) + f'(u(x,t))\, t = y(x,t) + f''(\theta)\, u\, t + O(u^2)\, t. \tag{18}$$

Since $u = O\Big(t^{-1/2}\Big)$ as $t \nearrow \infty$, $\theta = O\Big(t^{-1/2}\Big)$ and therefore

$$x = y(x,t) + f''(0)\, u\, t + O(u^2)\, t = y(x,t) + k\, u\, t + O(1). \tag{19}$$

This leads to

$$u(x,t) = \frac{x}{k\, t} + O(t^{-1}). \tag{20}$$

Combine with

$$s_+(t) \leqslant s_+ + \Big[\sqrt{2\, k\, q} + O\Big(t^{-1/2} \ln t\Big)\Big]\, t^{1/2}, \qquad s_-(t) \geqslant s_- - \Big[\sqrt{2\, k\, p} + O\Big(t^{-1/2} \ln t\Big)\Big]\, t^{1/2} \tag{21}$$

leads to the result.

For details see J. Smoller book pp. 295 – 297.

3. Convergence for periodic solutions.

When the initial value is periodic, we have better decay rate.

Theorem 2. *Let $f'' > 0$ and let $u_0 \in L^{\infty}(\mathbb{R})$ be piecewise monotonic periodic function of period p. Then we have*

$$|u(x,t) - \bar{u}_0| \leqslant \frac{2\,p}{h\,t} \tag{22}$$

where $\bar{u}_0 = \frac{1}{p}\int_0^p u_0(x)\,\mathrm{d}x$, and $h \equiv \min\left\{f''(u):\ |u| \leqslant \|u_0\|_{L^\infty}\right\}$.

The idea again is the consider backward characteristics. Since u_0 is piecewise monotonic and periodic, so is u at any time t. Let y_1, ..., y_n be the points where u changes from increasing to decreasing or vice versa, at time t. Then due to periodicity,

$$\text{Total variation of } u = 2\sum_{i \text{ odd}} u(y_i,t) - u(y_{i+1},t) = 2\sum_{i \text{ even}} u(y_i,t) - u(y_{i-1},t). \tag{23}$$

Here we have assumed that $[y_1, y_2], [y_3, y_4], \ldots$ are the decreasing intervals.

Now if we consider backward characteristics (which are straight lines!) starting from y_i's, we would have

$$p \geqslant \sum_{i \text{ odd}} (y_{i+1} - y_i) + \left[\sum_{i \text{ odd}} u(y_i,t) - u(y_{i+1},t)\right] t \geqslant \left[\sum_{i \text{ odd}} u(y_i,t) - u(y_{i+1},t)\right] t. \tag{24}$$

This shows the total variation of u decays like t^{-1}, and the desired result follows.

Further readings.

- J. Smoller, **Shock Waves and Reaction-Diffusion Equations**, §C of Chapter 16.