The Single Conservation Law: Existence and Uniqueness

In this lecture we will prove the existence and uniqueness of entropy solutions to the single conservation law

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0 \tag{1}$$

where $f'' > 0$.

- Many details of the proofs are omitted, because some of them are too hard while some others too easy. Anyone interested in these details should read Chapter 16 of Smoller's book.

1. Existence of entropy solution.

Theorem 1. *Let $u_0 \in L^\infty(\mathbb{R})$, and let $f \in C^2(\mathbb{R})$ with $f'' > 0$ on $\{u\colon |u| \leqslant \|u_0\|_{L^\infty}\}$. Then there exists a weak solution u with the following properties:*

a) $|u(x,t)| \leqslant \|u_0\|_{L^\infty} \equiv M$, $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

b) There is a constant $E > 0$, depending only on M, $\mu = \min\{f''(u)\colon |u| \leqslant \|u_0\|_{L^\infty}\}$ and $A = \max\{|f'(u)|\colon |u| \leqslant \|u_0\|_{L^\infty}\}$ such that for every $a > 0, t > 0$, and $x \in \mathbb{R}$,

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}. \tag{2}$$

c) u is stable and depends continuously on u_0 in the following sense: If $v_0 \in L^\infty(\mathbb{R})$ with $\|v_0\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}$, and v is the corresponding solution constructed from the process in the proof, then for every $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, and every $t > 0$,

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|\, \mathrm{d}x \leqslant \int_{x_1 - At}^{x_2 + At} |u_0(x) - v_0(x)|\, \mathrm{d}x. \tag{3}$$

Remark 2. We first check that a) – c) are satisfied by every C^1 solution. In particular, recall that when u is C^1, we have

$$u(x,t) = u_0(x_0) \tag{4}$$

where $x_0 + f'(u_0(x_0))\, t = x$. Thus

$$u(x,t) = u_0(x - f'(u(x,t))\, t). \tag{5}$$

Differentiating we obtain

$$u_x(x,t) = u_0'(x_0)\, [1 - f''(u)\, u_x\, t] \implies u_x(x,t) = \frac{u_0'(x_0)}{1 + u_0'(x_0)\, f''(u_0(x_0))\, t} \leqslant \frac{(\min f'')^{-1}}{t}. \tag{6}$$

Remark 3. Note that c) implies the uniqueness only for solutions obtained from the construction in the proof. This does not exclude the existence that other entropy solutions not obtained from the particular construction process.

Remark 4. The entropy condition implies some regularity in the solution. More specifically, recall that the "total variation" of a function f over an interval $[a,b]$ is defined as

$$\|f\|_{\mathrm{TV}} \equiv \sup_{\substack{n \in \mathbb{N} \\ a = a_1 < a_2 < \ldots < a_n = b}} \sum_{i=1}^{n-1} |f(a_i) - f(a_{i-1})|. \tag{7}$$

For any $t > 0$, take $c > E/t$, we see that $u(x,t) = v(x) + c\,x$ where $v(x)$ is decreasing. It is easy to verify that

$$\|u\|_{\mathrm{TV}} \leqslant \|v\|_{\mathrm{TV}} + \|c\,x\|_{\mathrm{TV}} < \infty \tag{8}$$

over any finite interval. Therefore although the initial data is only L^∞, the solution becomes more regular as soon as $t > 0$ in the sense that it has locally bounded total variation for each $t > 0$.

One can easily show that, a function with locally bounded total variation can have at most countably many jump discontinuities, and is differentiable almost everywhere.

Now we sketch the proof of the theorem. The idea is to consider the following finite difference discretization of the equation

$$\frac{u_n^{k+1}-\left(u_{n+1}^k+u_{n-1}^k\right)/2}{\Delta t}+\frac{f\left(u_{n+1}^k\right)-f\left(u_{n-1}^k\right)}{2\,\Delta x}=0 \tag{9}$$

with initial values

$$u_n^0 \equiv u_0(n\,\Delta x) \tag{10}$$

and show that

- The solution satisfies the discrete versions of a), b), c).
- As $\Delta t, \Delta x \searrow 0$, the discrete solutions (or a subsequence of them) converge to a weak solution satisfying a), b), c). The convergence would be a weak version of $u_n^k \to u(n\,\Delta x, k\,\Delta t)$.

1. The discrete solutions satisfy a) – c). Recall the notation $M \equiv \|u_0\|_{L^\infty}$, $\mu = \min\left\{f''(u)\colon |u| \leqslant \|u_0\|_{L^\infty}\right\}$ and $A = \max\left\{|f'(u)|\colon\ |u| \leqslant \|u_0\|_{L^\infty}\right\}$.

   a) Show that $\left|u_n^k\right| \leqslant M$ for all $n \in \mathbb{Z}, k \in \mathbb{Z}_+ \cup \{0\}$.

   Some manipulation of the difference equation gives

   $$u_n^{k+1}=\left(\frac{1}{2}+\frac{\Delta t}{2\,\Delta x}\,f'\left(\theta_n^k\right)\right)u_{n-1}^k+\left(\frac{1}{2}-\frac{\Delta t}{2\,\Delta x}\,f'\left(\theta_n^k\right)\right)u_{n+1}^k. \tag{11}$$

   This implies

   $$\left|u_n^{k+1}\right| \leqslant \max\left\{\left|u_{n-1}^k\right|, \left|u_{n+1}^k\right|\right\} \tag{12}$$

   as long as the CFL condition

   $$\frac{A\,\Delta t}{\Delta x} \leqslant 1 \tag{13}$$

   is satisfied.

   b) Let

   $$E \equiv \min\left(\mu/2, A/4\,M\right)^{-1}. \tag{14}$$

   We will show that

   $$\frac{u_n^k-u_{n-2}^k}{2\,\Delta x} \leqslant \frac{E}{k\,\Delta t}. \tag{15}$$

   Note that this implies

   $$\frac{u_n^k-u_{n-2m}^k}{2\,m\,\Delta x} \leqslant \frac{E}{k\,\Delta t} \tag{16}$$

   which is the discrete version of the entropy condition.

   The proof is too long to be included here. Details can be found in J. Smoller's book, pp. 269 – 272.

   c) Let $\left\{u_n^k\right\}$ and $\left\{v_n^k\right\}$ be solutions corresponding to the initial values $\left\{u_n^0\right\}$ and $\left\{v_n^0\right\}$, respectively, with $\sup\left|u_n^0\right|, \sup\left|v_n^0\right| \leqslant M$. Then for $k > 0$ we have

   $$\sum_{|n| \leqslant N}\left|u_n^k-v_n^k\right|\Delta x \leqslant \sum_{|n| \leqslant N+k}\left|u_k^0-v_k^0\right|\Delta x. \tag{17}$$

   The proof of this is very similar to that of a) after setting $w_n^k \equiv u_n^k - v_n^k$ and representing w_n^{k+1} by w_{n-1}^k and w_{n+1}^k.

2. Convergence to entropy solutions

   - Preparations – uniform bounds.

To be able to pass to the limit, we need the following bounds (proofs are omitted. See Smoller's book for details):

a. L^∞ bound. This has already been established in the first step;

b. TV bound. We have

$$\sum_{|n|<X/\Delta x} \left|u_{n+2}^k - u_n^k\right| \leqslant c \tag{18}$$

for any $X > 0$ and $k\,\Delta t \geqslant \alpha > 0$. The RHS c depends on X and α.

c. L^1 local Lipschitz continuity in t. We have

Lemma. *If $\Delta t/\Delta x \geqslant \delta > 0$, and $\Delta t, \Delta x \leqslant 1$, then there exists an $L > 0$, independent of $\Delta t, \Delta x$, such that if $k > p$, $k - p$ is even and $p\,\Delta t \geqslant \alpha > 0$,*

$$\sum_{|n|\leqslant X/\Delta x} \left|u_n^k - u_n^p\right| \Delta x \leqslant L\,(k-p)\,\Delta t. \tag{19}$$

A similar estimate for $k - p$ odd also holds.

Note that the above is the discrete version of the L^1 local Lipschitz continuity in t:

$$\|u(x,t) - u(x,t')\|_{L^1(-X,X)} \leqslant C\,|t - t'|. \tag{20}$$

The reason why $k - p$ is even or odd matters is that u_n^{k+1} depends on u_n^{k-1} but not u_n^k.

- The limit exists.

The first thing we need to settle is in what sense are we talking about the convergence, as for each $(\Delta t,\, \Delta x)$ $\left\{u_n^k\right\}$ is just a bunch of numbers, while the alleged limit is a function. To fix this, we define a function $U_{\Delta t,\Delta x}$ out of each $\left\{u_n^k\right\}$ as follows:

$$U_{\Delta t,\Delta x}(x,t) = u_n^k \qquad \text{for all } n\,\Delta x \leqslant x < (n+1)\,\Delta x,\ k\,\Delta t \leqslant t < (k+1)\,\Delta t. \tag{21}$$

We will now show that there is a subsequence $U_i(x,\, t)$ which is a Cauchy sequence in L^1. More specifically, we show

Lemma. *There exists a subsequence $\{U_i\} \subset \{U_{\Delta t,\Delta x}\}$ which converges to a measurable function $u(x,t)$ in the sense that for any $X > 0$, $t > 0$, and $T > 0$, both*

$$\int_{|x|\leqslant X} |U_i(x,t) - u(x,t)|\,\mathrm{d}x \longrightarrow 0 \tag{22}$$

and

$$\int_{0\leqslant t\leqslant T} \int_{|x|\leqslant X} |U_i(x,t) - u(x,t)|\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0. \tag{23}$$

Proof. The main steps are the following.

a. Fix any $k > 0$. Since $U_{\Delta t,\Delta x}(\,\cdot\,,\, k\,\Delta t)$'s are uniformly bounded and each $U_{\Delta t,\Delta x}$ has locally finite total variation and therefore can be written as the difference of two monotone functions, we can find a subsequence converging pointwise according to a theorem obtained by E. Helley in 1912.[1]

1. The "Helley Selection Theorem" says, every sequence of uniformly bounded sequence of monotone functions contains a subsequence which converges at every point.

To see this, note that first by the standard diagonal argument we can find a subsequence (all subsequences will still be denoted $\{f_n\}$) which converges on a countable dense subset of $\mathbb{R}$, say $\mathbb{Q}$. Denote the limit by f. Now we extend the definition of f from $\mathbb{Q}$ to the whole $\mathbb{R}$ by setting $f(x) = \sup_{r\in\mathbb{Q},r\leqslant x} f(r)$ (wlog we can assume all f_n are non-decreasing, thus f is non-decreasing over $\mathbb{Q}$). Such $f(x)$ is non-decreasing and therefore can have only countably many discontinuities. We choose a subsequence once more to obtain f_n such that $\{f_n(x)\}$ is Cauchy for all $x \in \mathbb{Q}$ and for all x which is a discontinuity of f. Finally, at any continuous point of f, $f_n(x) \longrightarrow f(x)$ by the standard $\varepsilon/3$ argument.

b. For this subsequence, Lebesgue's dominated convergence theorem implies $U_i(x,t)$ converges in L^1 for this particular $t = k\,\Delta t$.

c. By a diagonal argument, we can obtain a new subsequence $U_i(x,t)$ which converges in L^1 for all $t \in \mathbb{Q}_+$.

d. By the standard $\varepsilon/3$-type argument, we can show that $U_i(x,t)$ converges in $L^1(\mathbb{R})$ for every t. Denote the limit by $u(x,t)$.

e. Since U_i is uniformly bounded, so is u, and therefore $\|U_i - u\|_{L^1(-X,X)}$ is a bounded function in t. Application of Lebesgue's dominated convergence theorem then gives

$$\int_{0 \leqslant t \leqslant T} \int_{|x| \leqslant X} |U_i(x,t) - u(x,t)|\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0. \tag{24}$$

□

- The limit satisfies the entropy condition.

  This follows from straightforward computation of

$$\frac{U_i(x,t) - U_i(x',t)}{x - x'} \tag{25}$$

  combined with the pointwise convergence.

- The limit is a weak solution.

  a. First as $u_0 \in L^\infty$ and therefore locally L^1, $U_i(x,0) \to u_0$ in L^1.

  b. In this last step, we take any $\phi \in C_0^3$, multiply the difference equation by $\phi_n^k\,\Delta x\,\Delta t$ where $\phi_n^k = \phi(n\,\Delta x, k\,\Delta t)$, and then sum over n, k. It is easy to obtain

$$\lim_{n,k \nearrow \infty} \iint [U_i \phi_t + f(U_i)\,\phi_x] + \int_{t=0} U_i\,\phi = 0 \tag{26}$$

  which immediately gives the same equality for u.

2. Uniqueness of entropy solution.

To show uniqueness, we need to show that if u, v are two entropy solutions, then necessarily $u - v = 0$ for almost all (x,t). Since $u, v \in L^1$, we only need to show that $u^\varepsilon - v^\varepsilon = 0$ for all ε where $u^\varepsilon = \frac{1}{\varepsilon}\,\rho\left(\frac{x}{\varepsilon}\right) * u$ is the mollified function from u. As u^ε and v^ε are smooth, we finally see that the only thing need to be shown is

$$\int (u - v)\,\phi = 0 \tag{27}$$

for all ϕ in, say, C_0^1.

- Main idea of the proof.

  What we have is the weak formulation which is satisfied by both u and v:

$$\iint u\,\psi_t + f(u)\,\psi_x + \int_{t=0} u_0\,\psi = 0, \quad \iint v\,\psi_t + f(v)\,\psi_x + \int_{t=0} v_0\,\psi = 0 \tag{28}$$

  where ψ is any C_0^1 function.

  Subtracting the two equations, and remembering $u_0 = v_0$, we have

$$\iint (u - v)\left[\psi_t + \frac{f(u) - f(v)}{u - v}\,\psi_x\right] = 0. \tag{29}$$

  Now setting $F(x,t) \equiv \frac{f(u) - f(v)}{u - v}$, all we need to do is to show that for any $\phi \in C_0^1$, we can find $\psi \in C_0^1$ such that

$$\psi_t + F(x,t)\,\psi_x = \phi. \tag{30}$$

The f_n constructed converges everywhere, the limit is f at all rational points as well as all points where f is continuous, at the remaining points, the limit may be other values than $f(x)$.

For initial conditions, we assume $\phi=0$ for $t>T$ and take $\psi=0$ along $t=T$.

- Where's the catch.

  The above transport equation can be solved (formally) by the method of characteristics. Let $x(t)$ solves
  $$\frac{\mathrm{d}x}{\mathrm{d}t}=F(x(t),t), \tag{31}$$
  we obtain
  $$\frac{\mathrm{d}\psi}{\mathrm{d}t}(x(t),t)=\phi(x(t),t), \qquad \psi\,|_{t=T}=0 \tag{32}$$
  which leads to
  $$\psi(x(t),t)=\int_T^t \phi(x(s),s)\,\mathrm{d}s. \tag{33}$$
  Remember that we want $\psi\in C^1$ and at the same time has compact support.

  - Does ψ have compact support?

    Recall first that we solve ψ by setting $\psi=0$ for $t\geqslant T$. Next notice that ψ can be non-zero only along those characteristics which passes the support of ϕ. Now since F is uniformly bounded, the slope of the characteristics are uniformly bounded and the boundedness of ψ's support follows.

  - Is $\psi\in C^1$?

    This is where the catch is. As u,v are only in L^1, $F(x,t)$ is in general not Lipschitz and therefore the characteristics may collide with one another. When that happens, ψ is not in C^1 anymore. In fact, as $F(x,t)$ can only be expected to be in $L^1\cap L^\infty$, even the existence of the solution is questionable!

- Fixing the problem.

  We have seen that the obstacle is that F is not Lipschitz. To overcome this, we replace F by a smooth approximation F^ε such that $F^\varepsilon\to F$ locally in L^1, and call the corresponding solution ψ^ε. Then we have
  $$\int\int (u-v)\,\phi=\int\int (u-v)\,[\psi_t^\varepsilon+F^\varepsilon(x,t)\,\psi_x^\varepsilon]\,\mathrm{d}x\,\mathrm{d}t. \tag{34}$$
  Comparing with the definition of weak solutions, we obtain
  $$\int\int (u-v)\,\phi=\int\int (u-v)\,[F(x,t)-F^\varepsilon(x,t)]\,\psi_x^\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{35}$$
  As soon as we have shown the uniform boundedness of ψ_x^ε, we can take $\varepsilon\searrow 0$ and obtain
  $$\int\int (u-v)\,\phi=0 \tag{36}$$
  and finish the proof.

- Uniform boundedness of ψ_x^ε.

  If we naïvely mollify F, there is no way we can obtain this bound as in general ψ_x^ε grows as the Lipschitz constant of F^ε grows, and the latter grows like ε^{-1}. On the other hand, the subtle situation here is that we can make F smooth in a more sophisticated manner, which allows us to take advantage of the entropy condition (which hasn't been used so far!).

  Instead of mollifying F directly, we mollify u,v and define
  $$F^\varepsilon(x,t)=\frac{f(u^\varepsilon)-f(v^\varepsilon)}{u^\varepsilon-v^\varepsilon}=\int_0^1 f'(\theta\,u^\varepsilon+(1-\theta)\,v^\varepsilon)\,\mathrm{d}\theta. \tag{37}$$
  This gives
  $$\frac{\partial F^\varepsilon}{\partial x}=\int_0^1 f''(\theta\,u^\varepsilon+(1-\theta)\,v^\varepsilon)\left[\theta\,\frac{\partial u^\varepsilon}{\partial x}+(1-\theta)\,\frac{\partial v^\varepsilon}{\partial x}\right]\mathrm{d}\theta \tag{38}$$
  which is uniformly bounded from above if $\frac{\partial u^\varepsilon}{\partial x}$ and $\frac{\partial v^\varepsilon}{\partial x}$ are so – and this is indeed the case due to the entropy condition.

Thus we obtain

$$\frac{\partial F^{\varepsilon}}{\partial x} \leqslant \frac{C}{t} \tag{39}$$

for some positive constant C.[2]

From this bound one can show that

$$|\psi_x^{\varepsilon}(x,t)| \leqslant C \log t^{-1} \tag{40}$$

using the argument presented on p. 287 of J. Smoller's book.

Finally, recall that we want to prove

$$\iint_{t \geqslant 0} (u-v)\,[F-F^{\varepsilon}]\,|\psi_x^{\varepsilon}|\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0 \tag{41}$$

as $\varepsilon \rightarrow 0$. It is easy see that $F - F^{\varepsilon} \rightarrow 0$ in L^1. Combine this with the fact tat $F - F^{\varepsilon}$ is uniformly bounded, we can show that

$$F^{\varepsilon} \longrightarrow F \qquad \text{in } L^p \text{ for any } 1 \leqslant p < \infty. \tag{42}$$

Now the desired limit holds as $|\psi_x^{\varepsilon}|$ is uniformly bounded for any $1 \leqslant q < \infty$.[3]

Further readings.

Missing details can be found in

- J. Smoller, **Shock Waves and Reaction-Diffusion Equations**, §A, §B of Chapter 16.

Exercises.

Exercise 1. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be of locally bounded total variation, that is the total variation over any finite interval is finite. Prove that

1. f can have at most countably many jump discontinuities. (Hint: Consider the number of jumps of size $> 1/n$ over the interval $[-n, n]$ and then sum up.)
2. f is differentiable almost everywhere. (Hint: Define $g(x) =$ The total variation of f over $[-x, x]$, show that both g and $f - g$ are monotone. The fact that monotone functions are differentiable almost everywhere can be taken for granted.)

Exercise 2. The most "natural" finite difference discretization of the equation is obviously

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} + \frac{f\left(u_{n+1}^k\right) - f\left(u_n^k\right)}{\Delta x} = 0. \tag{43}$$

Try to carry out the existence proof using this scheme. At which step does it break down?

2. Note that if we solve a transport equation forward, $u_t + a(x,t)\,u_x = \phi$, then when a is increasing, that is $a_x \geqslant 0$, the characteristics are moving away from one another, which means u_x remain bounded; In the current situation, we are solving backwards from $t = T$ to $t = 0$, thus $a_x \leqslant 0$ is "good" and $a_x > 0$ is "bad". This is why we do not need a lower bound of $\frac{\partial F^{\varepsilon}}{\partial x}$.

3. For an elementary – and therefore much more tricky – argument, see pp. 287 – 290 of J. Smoller's book.