Discontinuous Solutions of Conservation Laws

In this lecture we study 1D conservation laws:

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0 \qquad x \in \mathbb{R},\ t > 0. \tag{1}$$

Here $\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f_1(\boldsymbol{u}) \\ \vdots \\ f_n(\boldsymbol{u}) \end{pmatrix}$. Such a system is called conservation laws as it is derived from the following relations characterizing the change of $u_1, \ldots, u_n$ in any interval (a, b):

$$\int_a^b u_i(x,t)\,\mathrm{d}x = f_i(u_1(a,t), \ldots, u_n(a,t)) - f_i(u_1(b,t), \ldots, u_n(b,t)). \tag{2}$$

Thus the change of u_i is due to "fluxes" at the two ends of the interval.

For such problems, it is natural to consider the initial value problem, where we start from $\boldsymbol{u} = \boldsymbol{u}_0$.

1. Discontinuous solutions.

The paradigm example in conservation laws is the following Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{3}$$

Facing such an equation, the first thing one would like to do is to "simplify" it to

$$u_t + u\,u_x = 0 \tag{4}$$

and apply the method of characteristics as we did for the transport equation $u_t + a(x,t)\,u_x = 0$. Let's see what happens.[1]

Consider the curves

$$(x(t), t):\ \frac{\mathrm{d}x}{\mathrm{d}t} = u(x,t);\ x(0) = x_0. \tag{5}$$

Along each such curve, the equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x(t),t) = 0 \Longrightarrow u(x(t),t) = u(x(0),0) = u_0(x_0). \tag{6}$$

Now put this information back to the characteristics equation, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_0(x_0). \tag{7}$$

Summary: the characteristic curves starting from x_0 is a straight ray $\dot{x} = u_0(x_0)$, and $u = u_0(x_0)$ along this particular ray. This is true for all C^1 solutions.

However, checking a few examples reveals that if $u_0(x_0) > u_0(x_1)$ for $x_0 < x_1$, the solution cannot exist for all time.

Now we determine the largest time for which u stays in C^1. For initial value u_0, the solution is

$$u(x,t) = u_0(x_0),\ x = x_0 + u_0(x_0)\,t \Longrightarrow u(x,t) = u_0(x - u_0(x_0)\,t) = u_0(x - u(x,t)\,t). \tag{8}$$

from the method of characteristics. We need to determine the largest time for u to stay in C^1. Differentiating, we have

$$\partial_x u = u_0'(x - u(x,t)\,t)\,[1 - t\,\partial_x u], \qquad \partial_t u = u_0'(x - u(x,t)\,t)\,[-u(x,t) - t\,\partial_t u] \tag{9}$$

which gives (recall $u(x,t) = u_0(x_0)$ and $x - u(x,t)\,t = x_0$)

$$\partial_x u(x,t) = \frac{u_0'(x_0)}{1 + t\,u_0'(x_0)}, \qquad \partial_t u(x,t) = -\frac{u_0'(x_0)\,u_0}{1 + t\,u_0'(x_0)}. \tag{10}$$

1. Since the equation is quasi-linear, we have to assume $u \in C^1$ a priori. Therefore what we did below is searching for C^1 solutions only. In other words, if the method of characteristics fails to find any solution, it doesn't mean that there is no non-C^1 solutions which make sense. This point will be clear soon.

We see that the maximum time for u to be in C^1 is

$$T = \begin{cases} \left[\max_x \left\{ -u_0'(x) \right\} \right]^{-1} & u_0'(x) < 0 \text{ for some } x \\ \infty & u_0'(x) \geqslant 0 \text{ everywhere} \end{cases} \tag{11}$$

Similar arguments can be applied to the general scalar conservation law

$$u_t + f(u)_x = 0 \tag{12}$$

where the characteristics are the rays

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(u(x_0)), \qquad x(0) = x_0. \tag{13}$$

One can show that the solution cannot stay in C^1 for all time when $u_0' f''(u_0) < 0$ for some x.

Remark 1. In the following, we will assume f to be convex, that is $f'' \geqslant 0$ everywhere.

From the above discussion, it is clear that in general C^1 solution cannot exist for all time. At this stage, one can choose from

1. accept this fact, and only consider solutions up to its maximum time of existence;
2. try to "generalize" the idea of solutions, define "weak" solutions which exists for all time.

Either one is a good choice, until we consider the very purpose of studying such equations. It turns out not only we cannot stop at C^1 solutions, we have to also consider discontinuous initial values!

Example 2. (The Shock Tube and Riemann's Problem in Gas Dynamics) A "shock tube" is a long, thin, cylindrical tube containing a gas separated by a thin membrane. The gas is at rest on both sides but the two sides are in different states. Now the question is, what happens when the membrane suddenly disappears?

The equations for gas dyanmics are

$$\begin{aligned} v_t - u_x &= 0 &(14)\\ u_t + p(v, S)_x &= 0 &(15)\\ S_t &= 0 &(16) \end{aligned}$$

where $v = \rho^{-1}$ with ρ the density, u is the velocity and S is the entropy. For the shock tube case, we need to deal with discontinuous initial data:

$$(v_0, u_0, S_0) = \begin{cases} (v_l, 0, S_l) & x < 0 \\ (v_r, 0, S_r) & x > 0 \end{cases}. \tag{17}$$

2. Weak solutions.

We would like to define "weak solutions" for the equation[2]

$$u_t + f(u)_x = 0, \qquad u(x, 0) = u_0(x). \tag{18}$$

As any meaningful definition of "weak solutions" should coincide with the classical definition when the solution is smooth enough, we multiply the equation by a C^1 test function ϕ and integrate by parts as if u is C^1.

$$\left[u_t + f(u)_x \right] \phi(x, t) = 0 \Longrightarrow -\iint_\Omega u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t + \oint_{\partial\Omega} \phi(x, t) \left[u\, n_t + f(u)\, n_x \right] \mathrm{d}S = 0. \tag{19}$$

If we take Ω to be the intersection of the support of ϕ and the half-plane $t > 0$, we obtain

$$\iint_{t>0} u\, \phi_t + f(u)\, \phi_x \,\mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} u_0\, \phi \,\mathrm{d}x = 0. \tag{20}$$

2. For simplicity we deal with the scalar case here. We should keep in mind that the "real-world" problems are mostly systems of conservation laws. For them weak solutions can be defined similarly.

Notice that u no longer needs to be C^1 to make the above integrals meaningful. For $\phi \in C^1$, the only requirement we should put on u is that both u, $f(u)$ are measures. In particular, it is OK for u to be piecewise continuous.

Definition 3. *u is called a weak solution of*

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0 \tag{21}$$

if

$$\int\int_{t>0} u\,\phi_t + f(u)\,\phi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0\,\phi\,\mathrm{d}x = 0. \tag{22}$$

holds for any $\phi \in C_0^1$.

One can easily show that if $u \in C^1$ is a weak solution, then it also solves the equation in the classical sense.

3. The jump condition.

We have seen that the integrals in the definition of weak solutions make perfect sense even when u is discontinuous. Now we try to gain some idea of what a weak solution would look like by considering piecewise C^1 solutions – that is, u has discontinuities along some curves but is C^1 everywhere else. It turns out that such curves must satisfy special conditions.

Consider one such curve, denote it by Γ. Let $\phi \in C_0^1$ be supported in a small ball centering on Γ. The ball is so small that it does not intersect with the x-axis and u is C^1 everywhere in the ball except along Γ.

Denote this ball by D, which is divided into two parts D_1, D_2 by Γ. As $\phi = 0$ along the x-axis, the definition of weak solutions becomes

$$\int\int_D u\,\phi_t + f(u)\,\phi_x\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{23}$$

We write the left hand side as $\int\int_{D_1} + \int\int_{D_2}$ and try to use integration by parts.

Since u is C^1 in D_1, D_2, we have

$$\int\int_{D_1} u\,\phi_t + f(u)\,\phi_x\,\mathrm{d}x\,\mathrm{d}t = -\int\int \left[u_t + f(u)_x\right]\phi\,\mathrm{d}x\,\mathrm{d}t + \oint_{\partial D_1} \left[u\,n_t + f(u)\,n_x\right]\phi\,\mathrm{d}S \tag{24}$$

Since u solves the equation in the classical sense in D_1 (see exercise) we have

$$\int\int_{D_1} u\,\phi_t + f(u)\,\phi_x\,\mathrm{d}x\,\mathrm{d}t = \oint_{\partial D_1} \left[u\,n_t + f(u)\,n_x\right]\phi\,\mathrm{d}S. \tag{25}$$

Similarly

$$\int\int_{D_2} u\,\phi_t + f(u)\,\phi_x\,\mathrm{d}x\,\mathrm{d}t = \oint_{\partial D_2} \left[u\,n_t + f(u)\,n_x\right]\phi\,\mathrm{d}S. \tag{26}$$

Since ϕ vanishes on ∂D_1 except along Γ, we finally obtain

$$\int_\Gamma \left[[u]\,n_t + [f(u)]\,n_x\right]\phi\,\mathrm{d}S = 0 \tag{27}$$

where $[u]$ is the "jump" of u across Γ.

Now let Γ be determined by $\frac{\mathrm{d}x}{\mathrm{d}t} = s(x,t)$. We have $\frac{n_t}{n_x} = -s$ which gives

$$\int_\Gamma \left[-s\,[u] + [f(u)]\right]\phi\,\mathrm{d}S = 0. \tag{28}$$

Due to the arbitrariness of ϕ, the weak solution must satisfy

$$[f(u)] = s\,[u]. \tag{29}$$

This is called the *jump condition*. On can also do the same analysis for systems of conservation laws and obtain

$$[\boldsymbol{f}(\boldsymbol{u})] = s\,[\boldsymbol{u}]. \tag{30}$$

In the special case of gas dynamics, this condition is referred to as *Rankine-Hugoniot condition*.[3]

From the above derivation one clearly sees that, if u is piecewise smooth (C^1), and satisfies the jump condition, then u is a weak solution.

Example 4. We illustrate how this condition can be used to determine weak solutions. Consider the Burgers equation. Here $f(u)=\frac{u^2}{2}$. The jump condition becomes

$$\frac{1}{2}\left[u^2\right]=s\,[u] \tag{31}$$

or

$$s=\frac{\left[u_l^2-u_r^2\right]}{2\,[u_l-u_r]}. \tag{32}$$

We can use this to determine the solution for the initial data

$$u_0(x)=\begin{cases}1 & x<0\\ 1-x & 0\leqslant x\leqslant 1\\ 0 & x>1\end{cases}. \tag{33}$$

Remark 5. Consider the following two equations

$$u_t+\left(\frac{u^2}{2}\right)_x=0 \qquad \text{and} \qquad \left(\frac{u^2}{2}\right)_t+\left(\frac{u^3}{3}\right)_x=0 \tag{34}$$

We see that they are equivalent for C^1 solutions but produce totally different discontinuous solutions as they lead to different jump conditions. This implies that classical solutions are in fact not "natural" for problems involving conservation laws.

4. Entropy conditions.

It turns out that weak solutions are in general not unique. For example, consider the Burgers equation with initial data $u_0(x)=\begin{cases}0 & x<0\\ 1 & x>0\end{cases}$, it turns out that both

$$u_1(x,t)=\begin{cases}0 & x<t/2\\ 1 & x>t/2\end{cases}, \qquad u_2(x,t)=\begin{cases}0 & x<0\\ x/t & 0<x<1\\ 1 & x>t\end{cases} \tag{35}$$

are weak solutions.

The fix to this situation is the introduction of the so-called "entropy" condition

$$\frac{u(x+a,t)-u(x,t)}{a}\leqslant\frac{E}{t}, \qquad \forall a>0, t>0 \tag{36}$$

where E is independent of x,t. A solution satisfying this entropy condition is called an "entropy solution". For an entropy solution, if it has a discontinuity, then necessarily $u_l>u_r$. Since we are considering the case $f''>0$, we always have

$$f'(u_l)>s>f'(u_r) \tag{37}$$

where s is the speed of the discontinuity (that is, the discontinuity is the curve $\frac{\mathrm{d}x}{\mathrm{d}t}=s(x,t)$).

Remark 6. If we draw the characteristics, the entropy condition requires characteristics to "meet" at the discontinuity instead of "emanating" from it. Physically speaking, each characteristic curve is a carrier of information, the requirement that they "meet" at any discontinuity is the same as saying information must decrease across any shocks. This is consistent with the Second Law of thermodynamics. This point of view helps in appreciating the following discussion on the irreversibility of entropy solutions.

3. According to L. Tartar (An Introduction to Navier-Stokes Equation and Oceanography, 2008) this is a misnomer. Such "jump conditions" are in fact first discovered by Stokes and Riemann.

Irreversibility.

Another justification of the entropy condition comes from the irreversibility of the resulting solutions. Consider again the Burgers equation. We will show that there are many entropy solutions which equals $u_1(x) \equiv \left\{ \begin{array}{ll} 0 & x > 1/2 \\ 1 & x < 1/2 \end{array} \right.$ at time $t = 1$.

Take any $\varepsilon \in [0, 1]$. Define $u_\varepsilon(x, t)$ by

$$\left\{ \begin{array}{ll} 1 & x < t - \varepsilon/2 \\ \dfrac{x - \varepsilon/2}{t - \varepsilon} & t - \varepsilon/2 < x < \varepsilon/2 \\ 0 & x > \varepsilon/2 \end{array} \right. \text{ for } t \leqslant \varepsilon \text{ and } \left\{ \begin{array}{ll} 1 & x < t/2 \\ 0 & x > t/2 \end{array} \right. \text{ for } t > \varepsilon \tag{38}$$

We see that $u_\varepsilon(x, t)$ are entropy solutions and furthermore $u_\varepsilon(x, 1) = u_1(x)$. In other words, by knowing the solution at $t = 1$, there is no way we can figure out its values at earlier times. Thus information decreases (or entropy increases) with time.

On the other hand, for the non-entropy solution

$$u(x, t) = \left\{ \begin{array}{ll} 0 & x < t/2 \\ 1 & x > t/2 \end{array} \right., \tag{39}$$

we can indeed trace back. To see this, assume that we know $u(x, 1)$. Then by setting $t' = 1 - t$, $x' = x$, and $u'(x', t') = -u(x, t)$, we see that u' satisfies

$$u'_{t'} + \left(\frac{u'^2}{2} \right)_{x'} = 0, \qquad u'(x', 0) = \left\{ \begin{array}{ll} 0 & x < 1/2 \\ -1 & x > 1/2 \end{array} \right.. \tag{40}$$

The method of characteristics together with the jump condition then gives a unique piecewise C^1 weak solution for $t' > 0$. In other words, we can determine u at $t < 1$ with the knowledge of $u(x, 1)$. This violates the Second Law of thermodynamics.

Further readings.

This lecture is based on

- J. Smoller, **Shock Waves and Reaction-Diffusion Equations**, Chap. 15.

Exercises.

Exercise 1. Consider the general scalar conservation law

$$u_t + f(u)_x = 0, \qquad u(x, 0) = u_0 \tag{41}$$

Write it as

$$u_t + f'(u)\, u_x = 0 \tag{42}$$

and use the method of characteristics to find out the maximum time for u to stay in C^1.

Exercise 2. Let u be a weak solution of the scalar conservation law. Show that if $u \in C^1(\Omega)$ for some domain Ω, then it is a classical solution in Ω, that is

$$u_t + f(u)_x = 0 \text{ for } (x, t) \in \Omega, \qquad u(x, 0) = u_0 \text{ when } (x, 0) \in \Omega. \tag{43}$$

Exercise 3. Consider the Burgers equation with initial data $u_0(x) = \left\{ \begin{array}{ll} 0 & x < 0 \\ 1 & x > 0 \end{array} \right.$. Show that the following two functions

$$u_1(x, t) = \left\{ \begin{array}{ll} 0 & x < t/2 \\ 1 & x > t/2 \end{array} \right., \qquad u_2(x, t) = \left\{ \begin{array}{ll} 0 & x < 0 \\ x/t & 0 < x < 1 \\ 1 & x > t \end{array} \right. \tag{44}$$

are both weak solutions to the problem.

Exercise 4. Consider the Burgers equation with initial data $u_0(x) = \left\{ \begin{array}{ll} 1 & x < 0 \\ 0 & x > 0 \end{array} \right.$. Prove that $u(x, t) = \left\{ \begin{array}{ll} 1 & x < t/2 \\ 0 & x > t/2 \end{array} \right.$ is a weak solution. Show that it is furthermore an entropy solution.