Wave Equations – Explicit Formulas

In this lecture we derive the representation formulas for the wave equation in the whole space:

$$\Box u \equiv u_{tt} - \triangle u = 0, \quad \mathbb{R}^n \times (0, \infty); \qquad u(x,0) = g(x), \qquad u_t(x,0) = h(x). \tag{1}$$

It turns out that the properties of the solutions depend on the dimension. More specifically, there are three cases: $n=1$, $n>1$ even; $n>1$ odd. We will discuss in detail the three representative cases: $n=1,2,3$ (the order is actually $n=1,3,2$, for reasons that will be clear soon).

1. $n=1$.

We consider the 1D wave equation

$$u_{tt} - u_{xx} = 0, \mathbb{R} \times (0, \infty); \qquad u(x,0) = g(x), \qquad u_t(x,0) = h(x). \tag{2}$$

This equation can be solved via the following change of variables:

$$\xi = x + t; \qquad \eta = x - t, \tag{3}$$

and to make things clearer we set $\tilde{u}(\xi, \eta) = u(x,t)$.

With this change of variable we compute

$$\begin{aligned} u_t &= \tilde{u}_\xi \xi_t + \tilde{u}_\eta \eta_t = \tilde{u}_\xi - \tilde{u}_\eta & (4)\\ u_{tt} &= \tilde{u}_{\xi\xi} - 2\,\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta} & (5)\\ u_x &= \tilde{u}_\xi + \tilde{u}_\eta & (6)\\ u_{xx} &= \tilde{u}_{\xi\xi} + 2\,\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta} & (7) \end{aligned}$$

Therefore

$$u_{tt} - u_{xx} = 0 \Longleftrightarrow \tilde{u}_{\xi\eta} = 0 \Longleftrightarrow \tilde{u}(\xi,\eta) = \phi(\xi) + \psi(\eta) \Longleftrightarrow u(x,t) = \phi(x+t) + \psi(x-t). \tag{8}$$

Now using the initial values we have

$$\phi(x) + \psi(x) = g(x); \qquad \phi'(x) - \psi'(x) = h(x) \tag{9}$$

which yields

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{10}$$

This is d'Alembert's formula.

Theorem 1. *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, define u by*

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{11}$$

Then

i. $u \in C^2(\mathbb{R} \times [0, \infty))$;

ii. $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0, \infty)$;

iii. u *takes the correct boundary values:*

$$\begin{aligned} \lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) &= g(x_0); & (12)\\ \lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) &= h(x_0). & (13) \end{aligned}$$

Proof. The proof is by direct calculation and is left as an exercise. □

Remark 2. It is easy to generalize the above theorem to the case

$$g \in C^k, \quad h \in C^{k-1} \quad \Longrightarrow \quad u \in C^k. \tag{14}$$

But in general u cannot be smoother. For example, consider the case $h = g'$, then $u(x, t) = g(x + t)$. It is clear that u cannot have better regularity than g.

Remark 3. As we have shown in the exercises after the lecture on distributions, the formula

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{15}$$

remains true even for distributional solutions of the 1D wave equation.

2. Spherical means and Euler-Poisson-Darboux equation.

The case $n \geqslant 2$ is much more complicated. The idea is the reduce the wave equation to a 1D equation which can be solved explicitly. The reduction is fulfilled through introducing the following auxiliary functions.

Let $u = u(x,t)$. We define at each $x \in \mathbb{R}^n$,

$$\begin{aligned} U(x;r,t) &\equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(w,t)\,\mathrm{d}S_w, &(16)\\ G(x;r) &\equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} g(w)\,\mathrm{d}S_w, &(17)\\ H(x;r) &\equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} h(w)\,\mathrm{d}S_w. &(18)\end{aligned}$$

Note that when u is continuous, we can recover u from U by taking $r \searrow 0$.

It turns out that $U(x;r,t)$ as a function of r and t satisfies a 1D equation (which can be further transformed in to a 1D wave equation in the first quadrant) and enjoys the same regularity as u.

Lemma 4. (Euler-Poisson-Darboux equation) *Fix $x \in \mathbb{R}^n$. Let $u(x,t) \in C^m$, $m \geqslant 2$ solves the wave equation. Then*

$$U(x;r,t) \equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(w,t)\,\mathrm{d}S_w \tag{19}$$

belongs to $C^m\big(\bar{\mathbb{R}}_+ \times [0,\infty)\big)$, and satisfies

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \quad \mathbb{R}_+ \times (0,\infty); \qquad U(r,0) = G(r), \quad U_t(r,0) = H(r). \tag{20}$$

Remark 5. Notice that $\partial_{rr} - \frac{n-1}{r}\partial_r$ is just $\triangle$ in $\mathbb{R}^n$ with radial symmetry.

Proof. Recall that

$$U_r(x;r,t) = \frac{r}{n} \frac{1}{|B_r(x)|} \int_{B_r(x)} \triangle_y u(y,t)\,\mathrm{d}y = \frac{1}{n\,\alpha(n)\,r^{n-1}} \int_{B_r(x)} \triangle_y u(y,t)\,\mathrm{d}y \tag{21}$$

This shows $U \in C^1$, and we can define $U_r(x;0,t) = 0$.

Differentiating w.r.t r again,

$$\begin{aligned} U_{rr}(x;r,t) &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{n\,\alpha(n)\,r^{n-1}} \int_{B_r(x)} \triangle_y u(y,t)\,\mathrm{d}y\right] \\ &= \frac{1-n}{n} \frac{1}{|B_r|} \int_{B_r(x)} \triangle_y u + \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} \triangle_y u. &(22)\end{aligned}$$

This shows $U \in C^2$ and also $U_{rr}(x;0,t)$ can be defined.

We further have

$$U_{tt}(x;r,t) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u_{tt} = -\frac{1}{|\partial B_r|} \int_{\partial B_r(x)} \triangle_y u \tag{23}$$

using the equation. $\square$

3. $n=3$, Kirchhoff's formula.

Let U, G, H be the spherical means. We set

$$\tilde{U}=rU, \quad \tilde{G}=rG, \quad \tilde{H}=rH. \tag{24}$$

Some calculation yields

$$\tilde{U}_{tt}-\tilde{U}_{rr}=0 \quad \mathbb{R}_+\times(0,\infty); \qquad \tilde{U}=\tilde{G}, \quad \tilde{U}_t=\tilde{H}. \tag{25}$$

Remark 6. Note that here we used the fact that $n=3$.

Thus we need to solve the wave equation in the first quadrant.

Example 7. Consider the wave equation in the first quadrant:

$$u_{tt}-u_{xx}=0, \quad x>0, t>0; \qquad u(x,0)=g(x), \quad u_t(x,0)=h(x), \quad u=0 \text{ for } x=0, t>0. \tag{26}$$

Let

$$\tilde{u}(x,t)=\begin{cases} u(x,t) & x>0 \\ -u(-x,t) & x<0 \end{cases} \tag{27}$$

and define similarly $\tilde{g}, \tilde{h}$. Then it is clear that $\tilde{u}$ solves the wave equation with initial values $\tilde{g}, \tilde{h}$. Thus we have

$$\tilde{u}(x,t)=\frac{1}{2}[\tilde{g}(x+t)+\tilde{g}(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}\tilde{h}(y)\,\mathrm{d}y. \tag{28}$$

Therefore the solution to the original problem is

$$u(x,t)=\begin{cases} \frac{1}{2}[g(x+t)+g(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}h(y)\,\mathrm{d}y & x\geqslant t\geqslant 0 \\ \frac{1}{2}[g(x+t)-g(t-x)]+\frac{1}{2}\int_{t-x}^{t+x}h(y)\,\mathrm{d}y & t\geqslant x\geqslant 0 \end{cases} \tag{29}$$

Now for our purpose, we only need the case $t\geqslant r$ (remember that finally we will let $r\searrow 0$ and recover u from U). In this case

$$\tilde{U}(x;r,t)=\frac{1}{2}\Big[\tilde{G}(r+t)-\tilde{G}(t-r)\Big]+\frac{1}{2}\int_{t-r}^{t+r}\tilde{H}(y)\,\mathrm{d}y. \tag{30}$$

We have

$$\begin{aligned} u(x,t) &= \lim_{r\searrow 0}\frac{\tilde{U}(x;r,t)}{r} \\ &= \tilde{G}'(t)+\tilde{H}(t) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{|\partial B_t|}\int_{\partial B_t(x)}g(w)\,\mathrm{d}S_w\right)+\frac{t}{|\partial B_t|}\int_{\partial B_t(x)}h(w)\,\mathrm{d}S_w. \end{aligned} \tag{31}$$

Further computation yields

$$u(x,t)=\frac{1}{|\partial B_t|}\int_{\partial B_t(x)}[t\,h(w)+g(w)+\nabla g(w)\cdot(w-x)]\,\mathrm{d}S_w \tag{32}$$

which is the Kirchhoff's formula.

4. n=2, Method of descent and Poisson's formula.

It is not possible to simplify as we did in the $n=3$ case. Instead, we use the so-called "method of descent", which treats the solution $u(x,t)$ of the 2D wave equation as a solution to the 3D equation. We set

$$\bar{u}(x_1,x_2,x_3,t)\equiv u(x_1,x_2,t). \tag{33}$$

and define $\bar{g}, \bar{h}$ similarly.

Using the Kirchhoff's formula we have

$$\begin{aligned} u(x,t) &= \bar{u}(\bar{x}, t) \\ &= \frac{1}{|\partial B_t(\bar{x})|}\int_{\partial B_t(\bar{x})} t\,\bar{h}(\bar{w}) + \bar{g}(\bar{w}) + \nabla_{\bar{x}}\bar{g}(\bar{w})\cdot(\bar{w}-\bar{x})\,\mathrm{d}S_{\bar{w}}. \end{aligned} \tag{34}$$

where $\bar{x} = (x, x_3)$ and $B_t(\bar{x})$ is the ball in $\mathbb{R}^3$.

From definitions of the variaous bar-ed functions, we have

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B_t(\bar{x})} t\,h(y) + g(y) + \nabla_y g(y)\cdot(y-x)\,\mathrm{d}S_{\bar{w}} \tag{35}$$

where $\bar{w} = \left(y, \pm\sqrt{t^2 - |y|^2}\right)$.

Finally, let $D_t(x)$ denote the ball in $\mathbb{R}^2$ centered at x with radius t, we have

$$\begin{aligned} u(x,t) &= \frac{2}{4\pi t^2}\int_{D_t(x)} \frac{t\,h(y) + g(y) + \nabla_y g(y)\cdot(y-x)}{\left(1 - \frac{|y-x|^2}{t^2}\right)^{1/2}}\,\mathrm{d}y \\ &= \frac{1}{2}\frac{1}{|D_t|}\int_{D_t(x)} \frac{t\,g(y) + t^2 h(y) + t\,\nabla g\cdot(y-x)}{\left(t^2 - |y-x|^2\right)^{1/2}}\,\mathrm{d}y. \end{aligned} \tag{36}$$

This is the Poisson's formula.

Remark 8. (Huygens' Principle) We notice that the behavior of the solutions for the 2D and 3D wave equations are drastically different. In 2D, $u(x, t)$ depends on initial data in the whole ball $D_t(x)$ while in 3D it only depends on the data on the boundary of the ball $B_t(x)$. Or equivalently, in 3D the effect of a vibration is only felt at the front of its propagation while in 2D it is felt forever after the front passed.[1] This is the so-called Huygens' principle.

Remark 9. For general n, we define

$$\tilde{U}(r,t) = \left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}U(x; r, t)\right) \tag{37}$$

and define $\tilde{G}, \tilde{H}$ accordingly. Some calculation yields the solution

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\frac{1}{|\partial B_t|}\int_{\partial B_t(x)} g\,\mathrm{d}S\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\frac{1}{|\partial B_t|}\int_{\partial B_t(x)} h\,\mathrm{d}S\right)\right] \tag{38}$$

for n odd, where $\gamma_n = 1\cdot 3\cdot\cdots\cdot(n-2)$. Then the method of descent yields

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\frac{t^n}{|B_t|}\int_{B_t}\frac{g(y)\,\mathrm{d}y}{\left(t^2 - |y-x|^2\right)^{1/2}}\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\frac{t^n}{|B_t|}\int_{B_t}\frac{h(y)\,\mathrm{d}y}{\left(t^2 - |y-x|^2\right)^{1/2}}\right)\right].$$

for n even, where $\gamma_n = 2\cdot 4\cdot\cdots\cdot(n-2)\cdot n$.

See L. C. Evans **Partial Differential Equations** pp. 75–80 for details.

Remark 10. (Nonhomogeneous problem) For the nonhomogeneous problem

$$\Box u = f, \qquad u = 0, \qquad u_t = 0, \tag{39}$$

we use the Duhamel's principle, obtaining

$$u(x,t) = \int_0^t u(x, t; s)\,\mathrm{d}s \tag{40}$$

1. If $u(x,t)$ also depends on data in the whole ball $B_t(x)$ in 3D, we would not be able to clearly hear anything!

where $u(x,t;s)$ solves

$$u_{tt} - \triangle u = 0, \qquad u(x,s;s) = 0, \quad u_t(x,s;s) = f(\,\cdot\,,s). \tag{41}$$

In particular, we have

- $n=1$:

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,\mathrm{d}y\,\mathrm{d}s. \tag{42}$$

- $n=3$:

$$u(x,t) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(y,t-|y-x|)}{|y-x|}\,\mathrm{d}y. \tag{43}$$

Here the integrand is called the "retarded potential".

Exercises.

Exercise 1. Prove the following theorem for 1D wave equations:

Theorem. *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, define u by*

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{44}$$

Then

i. $u \in C^2(\mathbb{R}\times[0,\infty))$*;*

ii. $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R}\times(0,\infty)$*;*

iii. u takes the correct boundary values:

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = g(x_0); \tag{45}$$

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) = h(x_0). \tag{46}$$

Exercise 2. (Equipartition of energy) Let $u \in C^2(\mathbb{R}\times[0,\infty))$ solve the initial-value problem for the 1D wave equation

$$u_{tt} - u_{xx} = 0, \quad u = g, \quad u_t = h; \tag{47}$$

Suppose g, h have compact support. Let

$$k(t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,\mathrm{d}x, \qquad p(t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \tag{48}$$

be the kinetic and potential energy. Prove that for t large enough, $k(t) = p(t) = \text{constant}$.

(Hint: Use d'Alembert's formula to compute the energy directly.)