Asymptotic Behavior and Energy Methods

In this lecture we first apply the maximum principles to study the asymptotic ($t \nearrow \infty$) behavior of the heat equation. Then we will introduce energy estimates for the heat equation.

1. Asymptotic behavior.

First recall that the solution to the initial value problem in $\mathbb{R}^n \times [0, \infty)$

$$u_t - \triangle u = 0 \qquad t > 0; \qquad u = g \quad t = 0 \tag{1}$$

is

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,\mathrm{d}y. \tag{2}$$

Now if g is integrable, we easily estimate

$$|u(x,t)| \leqslant \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |g(y)|\,\mathrm{d}y \searrow 0 \text{ as } t \nearrow \infty. \tag{3}$$

Note that the convergence is uniform.

Next we consider the following case

$$u_t - \triangle u = 0 \text{ in } \Omega \times (0, \infty); \qquad u(x,0) = f(x); \qquad u(x,t) = g(x) \text{ on } \partial\Omega \times (0, \infty). \tag{4}$$

As $t \nearrow \infty$, if $u \to u_\infty(x)$, then since u_∞ is independent of t, intuitively it should solve

$$\triangle u_\infty = 0 \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega. \tag{5}$$

We present some justification of this intuition in the following.

We first compute

$$\begin{aligned}(\partial_t - \triangle)\left(\frac{1}{2} u_t^2\right) &= u_t u_{tt} - u_t \triangle u_t - \sum_{i=1}^n u_{x_i t}^2 \\ &= u_t (u_t - \triangle u)_t - \sum_{i=1}^n u_{x_i t}^2 \\ &= -\sum_{i=1}^n u_{x_i t}^2 \leqslant 0. \end{aligned} \tag{6}$$

Therefore by the weak maximum principle,

$$\sup_{x \in \Omega} u_t^2 \tag{7}$$

is decreasing with time.

Decay of $\int u_t^2$.

Next we consider

$$E(t) \equiv \frac{1}{2} \int_\Omega |\nabla u|^2\,\mathrm{d}x. \tag{8}$$

Differentiating it we obtain

$$\dot{E}(t) = \int \nabla u \cdot \nabla u_t = -\int u_t \triangle u = -\int u_t^2\,\mathrm{d}x \leqslant 0. \tag{9}$$

where we have used the fact that $u_t = 0$ on $\partial\Omega$ at any time t.

Differentiating again, we have

$$\begin{aligned}\ddot{E}(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} \int u_t^2 \\ &= -\int \triangle\left(u_t^2\right)\mathrm{d}x + 2\int_\Omega |\nabla u_t|^2 \\ &= -\int_{\partial\Omega} \frac{\partial\left(u_t^2\right)}{\partial n}\,\mathrm{d}S + 2\int_\Omega |\nabla u_t|^2 \\ &\geqslant 0. \end{aligned} \tag{10}$$

Here the last inequality comes from the fact that $u_t^2 \geqslant 0$ in Ω and $= 0$ on $\partial\Omega$, therefore the outer-normal derivative has to be non-positive.

Thus we have $\dot{E}(t) \leqslant 0$ but is non-decreasing. This implies there is $A \leqslant 0$ such that

$$\lim_{t \nearrow \infty} \dot{E}(t) = A. \tag{11}$$

Now if $A < 0$, it is clear that $E(t) < 0$ for t large enough, a contradiction. Therefore we conclude $A = 0$. Recalling

$$\dot{E}(t) = -\int u_t^2 \, \mathrm{d}x \tag{12}$$

we see that

$$\int_\Omega u_t^2 \, \mathrm{d}x \longrightarrow 0 \qquad \text{as } t \nearrow \infty. \tag{13}$$

Uniform pointwise decay of u_t^2.

We can also obtain uniform decay of u_t^2.

To do this, we first extend $u_t^2(x, 0)$ from Ω to $\mathbb{R}^n$ by setting it to be 0 outside Ω. The result is a non-negative, continuous (recall that $u_t^2(x, 0) = 0$ on $\partial\Omega$), compactly supported function. Denote it by l. Now define

$$v(x, t) \equiv \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} l(y) \, \mathrm{d}y. \tag{14}$$

It is clear that v remains nonnegative and

$$v_t - \triangle v = 0. \tag{15}$$

Setting $w \equiv u_t^2 - v$, we have

$$w_t - \triangle w \leqslant 0 \text{ in } \Omega \times (0, \infty); \qquad w \leqslant 0 \text{ on } \partial\Omega \times (0, \infty); \qquad w = 0 \text{ on } \Omega \times \{t = 0\}. \tag{16}$$

Now weak maximum principle yields $w \leqslant 0$ for all time and thus

$$u_t^2 \leqslant v \longrightarrow 0 \qquad \text{as } t \nearrow \infty \tag{17}$$

pointwise.

Convergence to solutions of the Laplace equation.

If we know already that the problem

$$\triangle u_\infty = 0 \text{ in } \Omega; \qquad u_\infty = g \text{ on } \partial\Omega \tag{18}$$

has a solution, we have

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $g(x, t)$ be continuous on $\partial\Omega \times (0, \infty)$, and suppose*

$$\lim_{t \nearrow \infty} g(x, t) = g_\infty(x) \quad \text{uniformly in } x \in \partial\Omega \tag{19}$$

Let $F(x, t)$ be continuous on $\Omega \times (0, \infty)$, and suppose

$$\lim_{t \nearrow \infty} f(x, t) = f_\infty(x) \quad \text{uniformly in } x \in \Omega \tag{20}$$

Let $u(x, t)$ be a solution of

$$\triangle u - u_t = f \quad x \in \Omega, 0 < t < \infty; \qquad u = g \quad x \in \partial\Omega,\ 0 < t < \infty. \tag{21}$$

Let $v(x)$ be a solution of

$$\triangle u_\infty = f_\infty \quad x \in \Omega; \qquad v = g_\infty \quad x \in \partial\Omega, \tag{22}$$

Then

$$\lim_{t \nearrow \infty} u(x, t) = u_\infty(x) \tag{23}$$

uniformly in $x \in \Omega$.

Proof. Let $w=u-u_\infty$. Then clearly

$$\begin{aligned} \triangle w - w_t &= f - f_\infty \quad \text{in } \Omega\times(0,\infty) && (24)\\ w &= g - g_\infty \quad \text{on } \partial\Omega\times(0,\infty). && (25)\end{aligned}$$

We will show that there is a constant C such that for any $\varepsilon>0$, there is t_0 such that $\sup_x |w(x,t)| \leqslant C\varepsilon$.

Fix ε. Let τ be such that

$$\begin{aligned} \sup_{x\in\Omega} |f-f_\infty| &< \varepsilon \qquad t>\tau && (26)\\ \sup_{x\in\Omega} |g-g_\infty| &< \varepsilon \qquad t>\tau && (27)\end{aligned}$$

Denote

$$c_0 \equiv \sup_{x\in\Omega} |w(x,\tau)|. \tag{28}$$

Now choose $R>0$ such that $2\,x_1<R$ for all $(x_1,\ldots,x_n)\in\Omega$. Set

$$k(x)\equiv e^R - e^{x_1}. \tag{29}$$

Simple computation yields

$$\triangle k = -\,e^{x_1} \leqslant -\,\kappa \equiv -\inf_{x\in\Omega} e^{x_1}. \tag{30}$$

Now let

$$\kappa_0 \equiv \inf_{x\in\Omega} k(x); \qquad \kappa_1\equiv \sup_{x\in\Omega} k(x), \tag{31}$$

and set

$$m(x,t)\equiv \varepsilon\,\frac{k(x)}{\kappa} + \varepsilon\,\frac{k(x)}{\kappa_0} + c_0\,\frac{k(x)}{\kappa_0}\,e^{-\frac{\kappa}{\kappa_1}(t-\tau)}. \tag{32}$$

Calculation yields

$$\triangle m - m_t < -\,\varepsilon. \tag{33}$$

One further has

$$m(x,\tau) > c_0 \geqslant |f-f_\infty|(x,\tau) \qquad x\in\Omega \tag{34}$$

$$m(x,t) > \varepsilon \geqslant |g-g_\infty|(x,t) \qquad x\in\partial\Omega, \tau\leqslant t<\infty. \tag{35}$$

We apply weak maximum principle to $m(x,t)\pm w(x,t)$ on $\Omega\times[\tau,\infty)$ and obtain

$$|w(x,t)| \leqslant m(x,t) \leqslant \varepsilon\left(\frac{\kappa_1}{\kappa}+\frac{\kappa_1}{\kappa_0}\right) + c_0\,\frac{\kappa_1}{\kappa_0}\,e^{-\frac{\kappa}{\kappa_1}(t-\tau)}. \tag{36}$$

It is clear that one can choose t_0 such that the RHS is less than

$$\varepsilon\left(\frac{\kappa_1}{\kappa}+\frac{\kappa_1}{\kappa_0}+1\right) \tag{37}$$

for all $t>t_0$. Note that κ,κ_0,κ_1 are all independent of ε. □

2. Other estimates.

2.1. Fourier splitting.

In this section we introduce the Fourier spliting method introduced by Maria Schonbek in the late 1980s. It has proven to be an effective method in getting quantitative asymptotic behaviors for nonlinear equation involving the heat operator (for example, the Navier-Stokes equations and reaction-diffusion type equations).

The starting point is the energy inequality[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u^2\,\mathrm{d}x \leqslant -\int\, |\nabla u|^2\,\mathrm{d}x. \tag{38}$$

1. which is an equality in the case of the heat equation.

Recall Plancherel's theorem

$$\int f(x)^2\,\mathrm{d}x = \int \hat{f}(\xi)^2\,\mathrm{d}\xi \tag{39}$$

where $\hat{f}$ is the Fourier transform of f. Now taking the Fourier transform of the energy inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \hat{u}^2\,\mathrm{d}\xi \leqslant -C\int_{\mathbb{R}^n} |\xi|^2|\hat{u}|^2\,\mathrm{d}\xi \leqslant -C\int_{|\xi|>r} |\xi|^2|\hat{u}|^2\,\mathrm{d}\xi. \tag{40}$$

Taking $r = \left(\frac{n}{C\,(t+1)}\right)^{1/2}$ we obtain (this C is the particular C in the above estimate)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} |\hat{u}|^2\,\mathrm{d}\xi \leqslant -\frac{n}{t+1}\int_{|\xi|>r} |\hat{u}|^2\,\mathrm{d}\xi = -\frac{n}{t+1}\int_{\mathbb{R}^n} |\hat{u}|^2\,\mathrm{d}\xi + \frac{n}{t+1}\int_{|\xi|\leqslant r} |\hat{u}|^2\,\mathrm{d}\xi. \tag{41}$$

Multiply both sides by $(t+1)^n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(t+1)^n\int |\hat{u}|^2\,\mathrm{d}\xi\right] \leqslant n\,(t+1)^{n-1}\int_{|\xi|\leqslant r} |\hat{u}|^2\,\mathrm{d}\xi. \tag{42}$$

Now assume that

$$|\hat{u}| \leqslant A \qquad \text{for all } |\xi| \leqslant r. \tag{43}$$

We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(t+1)^n\int |\hat{u}|^2\,\mathrm{d}\xi\right] \leqslant n\,(t+1)^{n-1}\,r^n\,\alpha(n)\,A^2 = C^{-n/2}\,A^2\,n^{\frac{n}{2}+1}\,(t+1)^{n-1-\frac{n}{2}}. \tag{44}$$

Integrating this estimate gives

$$(t+1)^n\int |\hat{u}|^2\,\mathrm{d}\xi \leqslant \int |\hat{u}_0|^2\,\mathrm{d}\xi + c\,(t+1)^{\frac{n}{2}} \tag{45}$$

which gives

$$\int u^2\,\mathrm{d}x = \int |\hat{u}|^2\,\mathrm{d}\xi \leqslant c\,(t+1)^{-n/2}. \tag{46}$$

Remark 2. It is obvious that the above argument can be easily adapted to the case when the original energy inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u^2\,\mathrm{d}x \leqslant -C\int |\nabla^m u|^2\,\mathrm{d}x. \tag{47}$$

In this case we take $r = \left(\frac{n}{C\,(t+1)}\right)^{1/2m}$ and obtain

$$\int u^2\,\mathrm{d}x \leqslant c\,(t+1)^{-\frac{n}{2m}}. \tag{48}$$

Remark 3. It is clear that the key in the argument is the existence of a constant A such that

$$|\hat{u}| \leqslant A \qquad \text{for all } |\xi| \leqslant r = \left(\frac{n}{C\,(t+1)}\right)^{1/2}. \tag{49}$$

For the heat equation this is true since $|\hat{u}|$ actually decays with time. One can obtain the same bound for many other equations including conservation laws, Navier-Stokes equations, MHD equations, etc. See for example Maria Schonbek *The Fourier splitting method*, Advances in Geometric Analysis and Continuum Mechanics, (269-274), Internatl. Press, Cambridge, Ma. (1995)[2] for obtaining such bound for the n-dimensional scalar conservation law.

3. Energy estimates.

2. Available online at http://math.ucsc.edu/~schonbek/Publications/publications.html.

We use integration by parts to show uniqueness for the initial/boundary-value problem

$$u_t - \triangle u = f \text{ in } \Omega_T; \qquad u = g \text{ on } \partial^* \Omega_T. \tag{50}$$

We assume $\Omega \subset \mathbb{R}^n$ is open, bounded, and that $\partial\Omega$ is C^1. Let $T > 0$ be fixed.

Theorem 4. (Uniqueness) *There exists at most one solution in* $C_1^2(\bar{\Omega}_T)$.

Proof. If $\tilde{u}, u$ are two different solutions, we set $w = u - \tilde{u}$. Then w solves

$$w_t - \triangle w = 0 \text{ in } \Omega_T; \qquad w = 0 \text{ on } \partial^* \Omega_T. \tag{51}$$

Now set

$$e(t) \equiv \int_\Omega w^2(x,t)\,\mathrm{d}x. \tag{52}$$

It is clear that it suffices to show $e(t) \equiv 0$ for $0 \leqslant t \leqslant T$. We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^2(x,t)\,\mathrm{d}x \\ &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\big(w^2(x,t)\big)\,\mathrm{d}x \\ &= \int_\Omega 2\,w(x,t)\,w_t(x,t)\,\mathrm{d}x \\ &= 2\int_\Omega w(x,t)\,\triangle w(x,t)\,\mathrm{d}x \qquad \text{(We have used the equation here)} \\ &= -2\int_\Omega |\nabla w(x,t)|^2\,\mathrm{d}x \leqslant 0. \end{aligned} \tag{53}$$

Combined with $e(0) = 0$, we conclude that

$$e(t) \equiv 0 \tag{54}$$

for all $0 \leqslant t \leqslant T$ and ends the proof. □

4. Backward uniqueness.

An application of the energy methods is the following backward uniqueness result.

Theorem 5. (Backward uniqueness) *Let u and $\tilde{u}$ solve the heat equation in Ω_T with the same boundary conditions on $\partial\Omega \times [0,T]$ (that is, the initial values may be different). Then if*

$$u(x,T) = \tilde{u}(x,T) \qquad x \in \Omega \tag{55}$$

then

$$u \equiv \tilde{u} \qquad \text{in } \Omega_T. \tag{56}$$

Remark 6. We know from basic PDE courses (those solving the heat equation via Fourier expansion) that the inverse heat equation:

$$u_t - \triangle u = 0 \qquad \text{in } \Omega_T; \qquad u(x,T) = g(x) \tag{57}$$

is ill-posed, in the sense that if

$$\tilde{u}_t - \triangle \tilde{u} = 0 \qquad \text{in } \Omega_T; \qquad \tilde{u}(x,T) = \tilde{g}(x) \tag{58}$$

then the difference $\tilde{g} - g$ is magnified by an exponential factor, that is

$$|u - \tilde{u}|(t) \sim e^{C(T-t)}\,|g - \tilde{g}|. \tag{59}$$

However, a moment's inspection of this suggests that if $g = \tilde{g}$, $u = \tilde{u}$, which is just the backward uniqueness we are going to prove now.

Proof. Write $w = u - \tilde{u}$. Set

$$e(t) = \int_\Omega w^2(x,t)\,\mathrm{d}x. \tag{60}$$

Again it is clear that all we need to do is proving

$$e(t) \equiv 0 \qquad 0 \leqslant t \leqslant T \tag{61}$$

from $e(T) = 0$.

Recall that

$$\dot{e}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} e(t) = -2 \int_\Omega |\nabla w|^2\,\mathrm{d}x. \tag{62}$$

Differentiating once more, we obtain

$$\ddot{e}(t) \equiv \frac{\mathrm{d}^2}{\mathrm{d}t^2} e(t) = -4 \int \nabla w \cdot \nabla w_t = -4 \int \nabla w \cdot \nabla(\triangle w) = 4 \int (\triangle w)^2\,\mathrm{d}x. \tag{63}$$

Now since $w = 0$ on $\partial\Omega$, we have

$$\int_\Omega |\nabla w|^2\,\mathrm{d} = \int_\Omega \nabla w \cdot \nabla w = -\int_\Omega w \triangle w \leqslant \left(\int_\Omega w^2\right)^{1/2} \left(\int_\Omega (\triangle w)^2\right)^{1/2}. \tag{64}$$

Therefore

$$\dot{e}(t)^2 \leqslant e(t)\,\ddot{e}(t). \tag{65}$$

Now if $e(t) \not\equiv 0$, there exists (t_1, t_2) such that $e(t) > 0$ in (t_1, t_2) and $e(t_2) = 0$. We define

$$f(t) \equiv \log e(t) \qquad t_1 \leqslant t < t_2. \tag{66}$$

We calculate

$$\ddot{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{e}(t)}{e(t)}\right) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geqslant 0. \tag{67}$$

Thus f is a convex function, and as a consequence[3]

$$f((1-\alpha)\,t_1 + \alpha\,t) \leqslant (1-\alpha)\,f(t_1) + \alpha\,f(t). \tag{70}$$

for any $0 < \alpha < 1$ and $t_1 < t < t_2$. This gives

$$e((1-\alpha)\,t_1 + \alpha\,t) \leqslant e(t_1)^{1-\alpha}\,e(t)^{\alpha}. \tag{71}$$

Letting $t \nearrow t_2$ gives $e \equiv 0$. □

Exercises.

Exercise 1. Consider the following equation

$$u_t - a(x,t)\,u - \nu \triangle u = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{72}$$

where Ω is a bounded domain and $|a(x,\,t)| \leqslant A < \infty$ for all $x \in \Omega$. Using the energy estimate, show that there is a threshold $\nu_0 > 0$ (depending on A) such that there can be at most one solution in $C_1^2(\Omega_T)$ when $\nu \geqslant \nu_0$. (Hint: Poincaré's inequality is needed at one step of the proof.)

3. Note that $\ddot{f} \geqslant 0$ implies $\dot{f}(\xi) \geqslant \dot{f}(\eta)$ for any $\xi \geqslant \eta$. Using the mean value theorem, we have

$$\frac{f(t) - f((1-\alpha)\,t_1 + \alpha\,t)}{(1-\alpha)\,(t - t_1)} \geqslant \frac{f((1-\alpha)\,t_1 + \alpha\,t) - f(t_1)}{\alpha\,(t - t_1)} \tag{68}$$

which reduces to

$$f((1-\alpha)\,t_1 + \alpha\,t) \leqslant (1-\alpha)\,f(t_1) + \alpha\,f(t). \tag{69}$$