HEAT EQUATION – MAXIMUM PRINCIPLES

In this lecture we will discuss the maximum principles and uniqueness of solution for the heat equations.

1. Maximum principles.

The heat equation also enjoys maximum principles as the Laplace equation, but the details are slightly different. Recall that the domain under consideration is

$$\Omega_T \equiv \Omega \times [0, T); \qquad \partial^* \Omega_T \equiv \left(\bar{\Omega} \times \{0\}\right) \cup (\partial\Omega \times [0, T]). \tag{1}$$

We have the following strong maximum principle.

Theorem 1. (Maximum principles of the heat equation) *Assume $u \in C_1^2(\Omega_T) \cap C\left(\bar{\Omega}_T\right)$ solves*

$$u_t - \triangle u = 0 \tag{2}$$

in Ω_T.

i. **(Weak maximum principle)** *Then*

$$\max_{\bar{\Omega}_T} u = \max_{\partial^* \Omega_T} u. \tag{3}$$

ii. **(Strong maximum principle)** *Furthermore, if Ω is connected and there exists a point $(x_0, t_0) \in \Omega_T$ such that*

$$u(x_0, t_0) = \max_{\bar{\Omega}_T} u, \tag{4}$$

then u is constant in $\bar{\Omega}_{t_0}$.[1]

Remark 2. Intuitively, the maximum principles can be explained by the following observation. Recall that

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t - s)\, u(x, s)\, \mathrm{d}y \tag{5}$$

for any $s < t$, and the nonnegative function Φ satisfies

$$\int_{\mathbb{R}^n} \Phi(x - y, t - s)\, \mathrm{d}y = 1, \tag{6}$$

it is clear that $u(x, t)$ at time t is obtained from "averaging" $u(y, s)$ at an earlier time s, and therefore the maximum should be decreasing.

We present two proofs of this theorem. The first is through application of the maximum principle of the Laplace equation, the second is through establishing a "mean value" property. Note that the first method only yields the "weak maximum principle", that is the maximum inside is bounded by that on the boundary, instead of the "strong maximum principle", that is the maximum can only be attained at the boundary, unless the function is a constant.

Proof 1 – Maximum principles of the Laplace equation.

Theorem 3. (Weak maximum principle) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $u \in C_1^2(\Omega_T) \cap C^0\left(\bar{\Omega}_T\right)$, and satisfy*

$$u_t - \triangle u \leqslant 0 \tag{7}$$

in Ω_T, we then have

$$\sup_{\bar{\Omega}_T} u = \sup_{\partial^* \Omega_T} u. \tag{8}$$

---

1. Note that $\bar{\Omega}_{t_0}$ cannot be replaced by $\bar{\Omega}_T$, as can be seen by the following consideration. Let u solve the heat equation with initial value M (constant) and (side) boundary value $g(x) \leqslant M$. Now if $u_t \to 0$ as $t \searrow 0$, we can extend u to $t < 0$ by setting $u \equiv M$.

(If $T < \infty$, the supreme can be replaced by maximum).

Remark 4. The connection to the theory of Laplace equation can be seen by writing the inequality into

$$\triangle u \geqslant u_t. \tag{9}$$

Proof. If suffices to prove the conclusion for $\bar{\Omega}_{T'}$ for all $T' < T$. Fix one such T'. By compactness of $\bar{\Omega}_{T'}$ we know that the maximum is attained at some point (x_0, t_0).

1. First consider the case

$$\triangle u > u_t \qquad \text{in } \Omega_T. \tag{10}$$

   Now if $(x_0, t_0) \notin \partial^* \Omega_{T'}$, at (x_0, t_0), we have

   a) $u_t(t_0) \geqslant 0$ because otherwise $u(x_0, t_0 - \varepsilon) > u(x_0, t_0)$ for ε small enough;

   b) $\triangle u(x_0, t_0) \leqslant 0$ because x_0 maximizes u on $\Omega \times \{t_0\}$.

   But the two conclusions contradict each other.[2]

2. For the general case

$$\triangle u \geqslant u_t, \tag{11}$$

   we consider an auxiliary function

$$v(x, t) = u(x, t) - \varepsilon t. \tag{12}$$

   We easily check

$$\triangle v \geqslant v_t \tag{13}$$

   and can apply 1. Finally letting $\varepsilon \searrow 0$ we obtain the conclusion. □

From weak maximum principle one immediately obtains the uniqueness for nonhomogeneous initial/boundary-value heat equation when the domain Ω is bounded.

Theorem 5. (Weak maximum principle for $\Omega = \mathbb{R}^n$) *Suppose*

$$u_t - \triangle u \leqslant 0 \text{ in } \Omega_T; \qquad u(x, t) \leqslant M e^{\lambda |x|^2} \text{ in } \Omega_T \text{ for } M, \lambda > 0; \qquad u(x, 0) = g(x), \tag{14}$$

then

$$\sup_{\bar{\Omega}_T} u \leqslant \sup_{\mathbb{R}^n} g. \tag{15}$$

Remark 6. The reason why we need a bound on the growth rate at infinity is the following. Recall that the effect of the heat operator is to obtain solutions at later times from averaging solutions at earlier times via a kernel of the form $e^{-|x|^2/4}$. Note that the kernel decays at infinity as $e^{-c|x|^2}$. Therefore, if the solution grows faster than its inverse rate $e^{\lambda |x|^2}$, then it is possible to "input" extra stuff from infinity and thus violates uniqueness.

Here is an example of how one can "store" energy at $\pm\infty$ and then "input" it through averaging by the heat kernel, and thus obtain a nonzero solution with zero initial data.

Set

$$u(x, t) \equiv \sum_0^\infty \frac{g^{(n)}(t)}{(2n)!} x^{2n} \tag{16}$$

with

$$g(t) \equiv \begin{cases} e^{-\frac{1}{t^k}} & t > 0, k > 1 \\ 0 & t = 0. \end{cases} \tag{17}$$

2. The reason we use $T' < T$ here is to conclude $u_t \geqslant 0$ at the maximizer. Since the equation is solved on Ω_T only, u_t at $t = T$ is not known.

One can verify that u satisfies the equation by formally differentiating inside the summation.

With some calculation[3], one can further show that the series on the RHS is majorized by the Taylor expansion of

$$U(x,t)=\begin{cases}\exp\left[\frac{1}{t}\left(\frac{x^2}{\theta}-\frac{1}{2}t^{1-k}\right)\right] & t>0\\ 0 & t\leqslant 0\end{cases}. \tag{18}$$

for some $\theta>0$. Note that $U(x,t)$ cannot be bounded by $M\,e^{\lambda x^2}$ for any constants M,λ.

Proof. First we divide $(0,T)$ into subintervals with size $r<\frac{1}{4\,\lambda}$. It suffices to prove the claim on a subinterval. From now on we assume $T<\frac{1}{4\,\lambda}$.

Consider the auxiliary function

$$v(x,t)\equiv u(x,t)-\delta\,\frac{1}{(4\,\pi\,(T+\varepsilon-t))^{n/2}}\,e^{\left(\frac{|x|^2}{4(T+\varepsilon-t)}\right)}. \tag{19}$$

where ε is chosen such that $T+\varepsilon<\frac{1}{4\,\lambda}$.

It is easy to verify that

$$v_t-\triangle v\leqslant 0. \tag{20}$$

The strategy is the show first

$$v(x,t)\leqslant\sup_{\mathbb{R}^n} g \tag{21}$$

for any (x,t) and then letting $\delta\searrow 0$.

We first notice that, on the sphere $|x-y|=R$, we have

$$v(x,t)\leqslant M\,e^{\lambda(|x|+R)^2}-\delta\,\frac{1}{(4\,\pi\,(T+\varepsilon-t))^{n/2}}\,e^{\left(\frac{R^2}{4(T+\varepsilon-t)}\right)}\leqslant\sup_{\mathbb{R}^n} g \tag{22}$$

once R is big enough. Now apply the weak maximum principle on the domain $B_R\times(0,T)$, we obtain $v(x,t)\leqslant\sup g$. □

Proof 2 – Mean value property.

The mean value property for the heat equation turns out to be much more complicated than the one for the Laplace equation. We first define the "heat ball":

Definition 7. (Heat ball) *For fixed $x\in\mathbb{R}^n$, $t\in\mathbb{R}$, $r>0$, we define*

$$E(x,t;r)\equiv\left\{(y,s)\in\mathbb{R}^{n+1}\,|\,s\leqslant t,\ \Phi(x-y,t-s)\geqslant\frac{1}{r^n}\right\}. \tag{23}$$

Remark 8. We try to gain some idea of what $E(x,t;r)$ looks like. It is defined by

$$\frac{1}{(4\,\pi\,(t-s))^{n/2}}\,e^{-\frac{|x-y|^2}{4(t-s)}}\geqslant\frac{1}{r^n}. \tag{24}$$

Thus in the time direction, we have

$$t\geqslant s\geqslant t-\frac{r^2}{4\,\pi} \tag{25}$$

where the lower bound is from the fact that $e^{-\frac{|x-y|^2}{4(t-s)}}\leqslant 1$. Furthermore

$$E(x,t;r)\cap\{s=0\}=E(x,t;r)\cap\{s=t\}=\{x\}. \tag{26}$$

3. See pp. 211–213 in F. John **Partial Differential Equations**, 4ed. The constant θ is determined in Problem 3 on p. 73 of the same book.

Next, to find out the correct formula, we need to find a kernel $K(x-y,t-s)$ such that

$$\iint_{E(x,t;r)} K(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \tag{27}$$

is independent of r. Notice that

$$\frac{1}{(4\pi(t-s))^{n/2}}\,e^{-\frac{|x-y|^2}{4(t-s)}} \geqslant \frac{1}{r^n} \quad\Longleftrightarrow\quad \frac{1}{(4\pi(t'-s'))^{n/2}}\,e^{-\frac{|x'-y'|^2}{4(t'-s')}} \geqslant 1 \tag{28}$$

where

$$t'=t/r^2,\ \ s'=s/r^2,\ \ x'=x/r,\ \ y'=y/r. \tag{29}$$

Thus we have

$$\int_{E(x,t;r)} K(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s = \int_{E(x',t';1)} K(x-y,t-s)\,r^{n+2}\,\mathrm{d}y'\,\mathrm{d}s'. \tag{30}$$

This implies

$$K(x',t') = K(x,t)\,r^{n+2} \tag{31}$$

when

$$x'=x/r, \quad t'=t/r^2. \tag{32}$$

One can somehow[4] verify that [5]

$$\frac{1}{4}\int_{E(0,0;1)} \frac{|x|^2}{t^2}\,\mathrm{d}x\,\mathrm{d}t = 1. \tag{36}$$

Thus finally we have the correct formulation.

Theorem 9. (Mean value property for the heat equation) *Let $u\in C_1^2(\Omega_T)$ solve the heat equation, then*

$$u(x,t) = \frac{1}{4\,r^n}\iint_{E(x,t;r)} u(y,s)\,\frac{|x-y|^2}{(t-s)^2}\,\mathrm{d}y\,\mathrm{d}s. \tag{37}$$

for each $E(x,t;r)\subset\Omega_T$.

Proof. Without loss of generality we can set $(x,t)=(0,0)$ and denote $E(0,0;r)$ by E_r. Define

$$\phi(r) \equiv \frac{1}{r^n}\iint_{E_r} u(y,s)\,\frac{|y|^2}{s^2}\,\mathrm{d}y\,\mathrm{d}s = \iint_{E_1} u\big(r\,y', r^2\,s'\big)\,\frac{|y'|^2}{|s'|^2}\,\mathrm{d}y'\,\mathrm{d}s'. \tag{38}$$

In the following we will omit the prime and simply use y and y'.

---

4. I have no idea how to come up with this particular function.

5. As the computation is tedious and long, we put it in as a footnote and furthermore only present the main steps. First notice

$$E(0,0;1) \equiv \left\{\frac{1}{(4\pi t)^{n/2}}\,e^{\frac{|x|^2}{4t}} \geqslant 1\right\} = \left\{0\leqslant t\leqslant\frac{1}{4\pi};\, |x|^2\leqslant(2\,n\,t)\log\frac{1}{4\pi t}\right\}. \tag{33}$$

Thus the integral becomes

$$\int_0^{1/4\pi} t^{-2}\left[\int_{|x|^2\leqslant 2nt\log\frac{1}{4\pi t}} |x|^2\,\mathrm{d}x\right]\mathrm{d}t = \frac{n\,\pi^{n/2}}{n+2}\,\frac{2^{(n+2)/2}\,n^{(n+2)/2}}{\Gamma\left(\frac{n}{2}+1\right)}\int_0^{\frac{1}{4\pi}} t^{\frac{n-2}{n}}\left(\log\frac{1}{4\pi t}\right)^{\frac{n}{2}+1}\mathrm{d}t. \tag{34}$$

where we have used polar coordinates and the formula $\alpha(n)=\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ for the volume of n-dimension balls. Now setting $s=4\pi t$ and using the formulas

$$\lambda^{-z}\,\Gamma(z) = \int_0^1 t^{\lambda-1}\left(\log\frac{1}{t}\right)^{z-1}\mathrm{d}t; \qquad \Gamma(z+1) = z\,\Gamma(z), \tag{35}$$

we will see after careful calculations that most terms cancel out and what remains is 4.

thus we compute

$$\begin{aligned}\phi'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\iint_{E_1} u(r\,y, r^2\,s)\,\frac{|y|^2}{|s|^2}\,\mathrm{d}y\,\mathrm{d}s\right]\\ &= \iint \sum_{i=1}^{n} (\partial_{y_i}u)\,y_i\,\frac{|y|^2}{|s|^2} + 2\,r\,(\partial_s u)\,\frac{|y|^2}{s}\,\mathrm{d}y\,\mathrm{d}s\\ &= \frac{1}{r^{n+1}}\iint_{E_r} \sum_{i=1}^{n} (\partial_{y_i}u)\,y_i\,\frac{|y|^2}{|s|^2} + \frac{1}{r^{n+1}}\iint_{E_r} 2\,(\partial_s u)\,\frac{|y|^2}{s}\\ &\equiv A+B. \end{aligned}\tag{39}$$

Now we introduce

$$\psi \equiv -\frac{n}{2}\log(-4\,\pi\,s) + \frac{|y|^2}{s} + n\log r, \tag{40}$$

and note that $\psi = 0$ on ∂E_r.

Now we have

$$\begin{aligned} B &= \frac{1}{r^{n+1}}\iint_{E_r} 4\,(\partial_s u)\sum_{i=1}^{n} y_i\,(\partial_{y_i}\psi)\,\mathrm{d}y\,\mathrm{d}s\\ &= -\frac{1}{r^{n+1}}\iint_{E_r} 4\,n\,u_s\,\psi + 4\sum_{i=1}^{n} u_{s\,y_i}y_i\,\psi\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}\tag{41}$$

Now integrate by parts w.r.t. s, we have

$$\begin{aligned} B &= \frac{1}{r^{n+1}}\iint_{E_r} -4\,n\,u_s\,\psi + 4\sum_{i=1}^{n} u_{\,y_i}y_i\,\psi_s\,\mathrm{d}y\,\mathrm{d}s\\ &= \frac{1}{r^{n+1}}\iint_{E_r} -4\,n\,u_s\,\psi + 4\sum u_{y_i}\,y_i\left(-\frac{n}{2\,s} - \frac{|y|^2}{4\,s^2}\right)\mathrm{d}y\,\mathrm{d}s\\ &= \frac{1}{r^{n+1}}\iint_{E_r} -4\,n\,u_s\,\psi - \frac{2\,n}{s}\sum u_{y_i}\,y_i\,\mathrm{d}y\,\mathrm{d}s - A. \end{aligned}\tag{42}$$

Consequently, using the equation, we have

$$\begin{aligned} \phi'(r) &= A+B\\ &= \frac{1}{r^{n+1}}\iint_{E_r} -4\,n\,\triangle u\,\psi - \frac{2\,n}{s}\sum u_{y_i}\,y_i\,\mathrm{d}y\,\mathrm{d}s\\ &= \sum \frac{1}{r^{n+1}}\iint_{E_r} -4\,n\,u_{y_i}\,\psi_{y_i} - \frac{n}{2\,s}\,u_{y_i}\,y_i\,\mathrm{d}y\,\mathrm{d}s\\ &= 0. \end{aligned}\tag{43}$$

Thus we have

$$\phi(r) = \lim_{t\to 0}\phi(t) = u(0,0)\left(\lim_{t\to 0}\frac{1}{t^n}\iint_{E_t}\frac{|y|^2}{s^2}\,\mathrm{d}y\,\mathrm{d}s\right) = 4\,u(0,0). \tag{44}$$

□

From the mean value property we immediately have the strong maximum principle:

Proof. (of the strong maximum principle)

Suppose there is $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) = \max_{\bar{\Omega}_T} u$, then by picking r small enough so that $E(x_0, t_0; r) \subset \Omega_T$, and using the mean value property, we conclude that u is constant inside $E(x_0, t_0; r)$.

Next for any $(y_0, s_0) \in \Omega_T$ such that the line segment connecting x_0, y_0 is in Ω_T, we can show that $u(y_0, s_0) = u(x_0, t_0)$ whenever $s_0 < t_0$ by covering the line segment connecting (y_0, s_0) and (x_0, t_0) with the heat balls.

Finally, since Ω is connected, any y_0 can be connected from x_0 via finitely many line segments. And therefore $u(y, s) = u(x_0, t_0)$ for all $y \in \Omega$, $s < t_0$. □

Remark 10. It turns out there is also a version of mean value property involving only the surface integral over $\partial E(x, t; r)$. The idea is to compute the integral of $\left[\Phi(x - y, T - t) - r^{-n}\right](u_t - \triangle u)$ over $E(x, t; r)$ via integration by parts. This will give a representation formula for $u(x, t)$, which turns out to be

$$u(x,t) = \frac{1}{r^n} \int_{\partial E(x,t;r)} u(y,s) \frac{|x-y|}{2\,(t-s)} \,\mathrm{d}S. \tag{45}$$

See J. Jost **Partial Differential Equations**, pp. 81 – 83 for details.

The strong maximum principle can also be proved using this formula. See Theorem 4.1.3 on p.86 of J. Jost **Partial Differential Equations**.

2. Uniqueness.

As in the elliptic case, uniqueness is established through maximum principles.

Theorem 11. **(Uniqueness on bounded domains)** *Let $g \in C(\partial^* \Omega_T)$, $f \in C(\Omega_T)$. Then there exists at most one solution $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ of the initial/boundary value problem*

$$u_t - \triangle u = f \quad \text{in } \Omega_T; \qquad u = g \quad \text{on } \partial^* \Omega_T. \tag{46}$$

In the case $\Omega = \mathbb{R}^n$, we have to add extra condition.

Theorem 12. **(Uniqueness when $\Omega = \mathbb{R}^n$)** *Suppose $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0, T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ of the initial value problem*

$$u_t - \triangle u = f \text{ in } \mathbb{R}^n \times (0, T); \qquad u = g \text{ on } \mathbb{R}^n \times \{t = 0\} \tag{47}$$

satisfying the growth estimate

$$|u(x,t)| \leqslant M e^{\lambda |x|^2} \tag{48}$$

for constants $M, \lambda > 0$.

Remark 13. There are in fact infinitely many "physically incorrect" solutions, which do not satisfy the growth bound, to the zero initial value problem. See Chapter 7 of F. John **Partial Differential Equations**.

Exercises.

Exercise 1. Let $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ be a solution to

$$u_t - \triangle u = 0 \quad \text{in } \Omega_T; \qquad u = 0 \quad \text{on } \partial\Omega \times [0, T]; \qquad u = g \quad \text{on } \Omega \times \{t = 0\}, \tag{49}$$

where $g \geqslant 0$. Then u is positive everywhere within Ω_T if g is positive somewhere on Ω.