Sep. 24

HARMONIC FUNCTIONS

Since the Poisson equation

$$\triangle u = f \text{ in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{1}$$

is linear, the uniqueness of its solutions is equivalent to the uniqueness of the Laplace equation

$$\triangle u = 0 \text{ in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

That is, any harmonic function which vanishes on the boundary must be identically zero. However the theory of harmonic functions, or the study of the Laplace equation, has impact far beyond just establishing uniqueness for the Poisson equation. The major properties of harmonic equation – maximum principles, harnack inequalities, etc. – indeed guide the study of the whole class of elliptic PDEs. In this lecture we will try to study in detail the properties of harmonic functions.

Harmonic functions.

Definition 1. *A C^2 function satisfying $\triangle u = 0$ is called a harmonic function.*

1. Mean value formulas.

Theorem 2. *If $u \in C^2(\Omega)$ is harmonic, then*

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u \,\mathrm{d}x = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \,\mathrm{d}S. \tag{3}$$

for every ball $B_r(x) \subset \Omega$.

Proof. First note that since

$$\int_{B_r} u \,\mathrm{d}x = \int_0^r \left(\int_{\partial B_r} u \,\mathrm{d}S \right) \mathrm{d}r \tag{4}$$

we only need to prove

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \,\mathrm{d}S. \tag{5}$$

For simplicity we set $x = 0$ and use B_r to denote $B_r(0)$.

We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{|\partial B_r|} \int_{\partial B_r} u(y) \,\mathrm{d}S_y \right] &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{|\partial B_1|} \int_{\partial B_1} u(r\,w) \,\mathrm{d}S_w \right] \\ &= \frac{1}{|\partial B_1|} \int_{\partial B_1} \nabla u(r\,w) \cdot w \,\mathrm{d}S_w \\ &= \frac{1}{|\partial B_1|} \int_{\partial B_1} \nu \cdot \nabla u(r\,w) \,\mathrm{d}S_w \\ &= \frac{1}{|\partial B_r|} \int_{\partial B_r} \nu \cdot \nabla u(y) \,\mathrm{d}S_y \\ &= \frac{1}{|\partial B_r|} \int_{B_r} \triangle u(x) \,\mathrm{d}x = 0. \end{aligned} \tag{6}$$

The conclusion is obtained by taking $r \to 0$ (remember that u is continuous). □

The mean value formula is a particular property for harmonic functions and cease to be true for solutions to general elliptic equations,[1] but one can obtain other properties – maximum principles, harnack inequalities, gradient estimates – which are more stable and can be extended to more complicated, even nonlinear, equations. However, before discussing these properties, we first prove the converse of the above theorem, which – a bit surprisingly – claims that any continuous function satisfying the mean value formula is harmonic.

1. Nevertheless one can write down mean value formulas for regions other than balls. Functions satisfy those mean value formulas are solutions to elliptic equations too.

Remark 3. When $u\in C^2$, one easily reverses the argument to show the converse. What is surprising here is that only continuity is required.

Theorem 4. *If $u\in C(\Omega)$ satisfies*

$$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u\,\mathrm{d}S \tag{7}$$

for every $x\in\Omega$ and ball $B_r(x)\subset\Omega$, then u is C^∞ and is harmonic.

Proof. We only need to show $u\in C^\infty$ and the rest of the proof is trivial. To do this, we observe that for any radially symmetric function $\phi=\phi(r)$ supported in B_ε with $\int_{B_\varepsilon}\phi=1$, we have

$$u*\phi=u \tag{8}$$

for all $x\in\Omega$ such that $B_r(x)\subset\Omega$.

For simplicity we compute at $x=0$:

$$\begin{aligned}(u*\phi)(0) &= \int_{B_\varepsilon} u(y)\,\phi(-y)\,\mathrm{d}y\\ &= \int_{B_\varepsilon} u(y)\,\phi(y)\,\mathrm{d}y\\ &= \int_0^\varepsilon \left[\int_{\partial B_r} u(z)\,\phi(z)\,\mathrm{d}S_z\right]\mathrm{d}r\\ &= \int_0^\varepsilon \left[\int_{\partial B_r} u(z)\,\mathrm{d}S_z\right]\phi(r)\,\mathrm{d}r\\ &= \int_0^\varepsilon u(0)\,|\partial B_r|\,\phi(r)\,\mathrm{d}r\\ &= u(0)\int_{B_\varepsilon}\phi(r)\,\mathrm{d}x=u(0).\end{aligned} \tag{9}$$

Now take $\phi=\phi(r)\in C_0^\infty(B_1)$ and set $\phi_\varepsilon=\frac{1}{\varepsilon^n}\,\phi\left(\frac{x}{\varepsilon}\right)$. We see that for any $\Omega'\subset\subset\Omega$, there is $\varepsilon>0$ such that

$$u=u*\phi_\varepsilon\in C^\infty \qquad \text{on } \Omega' \tag{10}$$

Thus ends the proof. □

Remark 5. Inspection of the above proof reveals that the same conclusion holds when u is only assumed to be locally integrable.

The same method can be used to prove the following Weyl's lemma:

Lemma. (Weyl's lemma) *Let $u\colon\Omega\mapsto\mathbb{R}$ be measurable and locally integrable in Ω. Suppose that for all $\varphi\in C_0^\infty(\Omega)$,*

$$\int_\Omega u(x)\,\triangle\varphi(x)\,\mathrm{d}x=0. \tag{11}$$

then u is harmonic and, in particular, smooth. (In other words, the lemma claims that any locally integrable[2] distributional solution of the Laplace equation is smooth)

Proof. We sketch the key ideas.

1. Take a mollifier $\psi(x)$, let $\psi_\varepsilon(x)\equiv\varepsilon^{-n}\,\psi(x/\varepsilon)$. For any $\varphi\in C_0^\infty(\Omega)$, $\psi_\varepsilon*\varphi\in C_0^\infty(\Omega)$ for ε small enough. Using it as the test function, we obtain

$$\triangle(u*\psi_\varepsilon)=0. \tag{12}$$

in the classical sense.

2. *What conclusion can we make without local integrability?*

2. As a consequence,

$$(u*\psi_\varepsilon)(x)=\frac{1}{|B_r|}\int_{B_r(x)} u*\psi_\varepsilon\,\mathrm{d}x \tag{13}$$

for r small enough. Now let $\varepsilon\searrow 0$, since $u\in L^1$, the LHS $\to u$ in L^1 and therefore one can find a subsequence $\to u$ almost everywhere, while the RHS converges to $\int u\,\mathrm{d}x$. So u satisfies the mean value property. □

From the above results we see that in particular every harmonic function is C^∞. In fact we can show that they are actually analytic. These properties cease to look mysterious when we realize that after identifying $\mathbb{R}^2$ with $\mathbb{C}$, any harmonic function can be seen as the real (or imaginary) part of an analytic function. For example, Theorem 4 becomes natural now since any continuous complex function satisfying the mean value property must be analytic.

2. Maximum principles.

Theorem 6. (Strong maximum principle) *Suppose $u\in C^2(\Omega)\cap C(\overline{\Omega})$ is harmonic in Ω.*

a) Then

$$\max_{\bar{\Omega}} u=\max_{\partial\Omega} u. \tag{14}$$

b) If Ω is connected and there exists a point $x_0\in\Omega$ such that

$$u(x_0)=\max_{\bar{\Omega}} u, \tag{15}$$

then u is constant within Ω.

The claims remain true when max *is replaced by* min*.*

Proof.

a) This follows from b).

b) Assume there is $x_0\in\Omega$ such that

$$u(x_0)=\max_{\bar{\Omega}} u, \tag{16}$$

we try to show that u must be constant in the connect component containing x_0.

We define the subset

$$\Omega'\equiv\left\{x\in\Omega\,|\,u(x)=u(x_0)=\max_{\bar{\Omega}} u\right\}. \tag{17}$$

Then by continuity of u, Ω' is closed (relative to Ω).

On the other hand, the mean value formula implies that Ω' is also open (relative to Ω).

Now the connectedness of Ω forces $\Omega'=\Omega$. Thus ends the proof. □

Remark 7. From the maximum principle we immediately obtain the uniqueness of the solutions for the Dirichlet problem of the Poisson equation.

Using the maximum principle one can obtain local derivative estimates.

Theorem 8. (Derivative estimates) *Assume u is harmonic in Ω. Then*

$$|\partial^\alpha u(x_0)|\leqslant\frac{C_k}{r^{n+k}}\|u\|_{L^1(B_r(x_0))}{}^{3} \tag{18}$$

for each ball $B_r(x_0)\subset\Omega$ and each multi-index α of order $|\alpha|=k$ $(k=0,1,\ldots)$. Here

$$C_0=\frac{1}{\alpha(n)},\qquad C_k=\frac{\left(2^{n+1}\,n\,k\right)^k}{\alpha(n)}. \tag{19}$$

3. $\|u\|_{L^1(B_r(x_0))}\equiv\int_{B_r(x_0)}|u|\,\mathrm{d}x$.

Proof.

- $k=0$. Recall the mean value formula

$$u(x_0)=\frac{1}{|B_r|}\int_{B_r(x_0)} u(y)\,\mathrm{d}y=\frac{1}{r^n\,\alpha(n)}\int_{B_r(x_0)} u(y)\,\mathrm{d}y\leqslant\frac{1/\alpha(n)}{r^n}\,\|u\|_{L^1(B_r(x_0))}. \tag{20}$$

- $k=1$. First note that $\partial_{x_i}u$ is still harmonic. Therefore we can apply the mean value formula on $B_{r/2}(x_0)$:

$$\begin{aligned}|\partial_{x_i}u(x_0)| &= \left|\frac{1}{|B_{r/2}|}\int_{B_{r/2}(x_0)}\partial_{x_i}u(y)\,\mathrm{d}y\right| \\ &= \left|\frac{1}{|B_{r/2}|}\int_{\partial B_{r/2}(x_0)} n_i\,u(y)\,\mathrm{d}y\right| \\ &\leqslant \frac{1}{|B_{r/2}|}\,|\partial B_{r/2}|\sup_{\partial B_{r/2}(x_0)}|u(y)| \\ &\leqslant \frac{2\,n}{r}\sup_{\partial B_{r/2}(x_0)}|u(y)|.\end{aligned} \tag{21}$$

Now we apply the $k=0$ result to $|u(y)|$:

$$|u(y)|\leqslant\frac{1/\alpha(n)}{r^n}\int_{B_{r/2}(y)}|u|\leqslant\frac{1/\alpha(n)}{(r/2)^n}\int_{B_r(x_0)}|u|=\frac{2^n/\alpha(n)}{r^n}\,\|u\|_{L^1(B_r(x_0))}. \tag{22}$$

Substituting this back, we obtain

$$\begin{aligned}|\partial_{x_i}u(x_0)| &\leqslant \frac{2\,n}{r}\sup_{\partial B_{r/2}(x_0)}|u(y)| \\ &\leqslant \frac{2^{n+1}\,n}{\alpha(n)}\,\frac{1}{r^{n+1}}\|u\|_{L^1(B_r(x_0))}.\end{aligned} \tag{23}$$

- Induction. We assume the case $k-1$ has already been established.

  Let $|\alpha|=k$. Then $\partial^\alpha u=\partial_{x_i}(\partial^\beta u)$ where $|\beta|=k-1$ for some x_i. Now we can bound $\partial^\alpha u(x_0)$ by $\partial^\beta u(y)$ for $y\in\partial B_{r/k}(x_0)$, and then use the $k=1$ case on balls of radius $\frac{k-1}{k}r$. The details are left as an exercise. □

Remark 9. From the above derivative estimates it is easy to show that if u is harmonic in Ω then it is in fact not only C^∞ by analytic. See pp. 31–32 in L. C. Evans **Partial Differential Equations** for a proof.

3. Harnack inequalities.

An important property of elliptic equations is the Harnack inequality. More specifically, if $u\geqslant 0$ is harmonic in Ω then its osciallation inside Ω is controlled.

Theorem 10. (Harnack's inequality) *For any connected open set $\Omega'\subset\subset\Omega$*[4]*, there exists a positive constant C, depending only on Ω', such that*

$$\sup_{\Omega'}u\leqslant C\inf_{\Omega'}u \tag{24}$$

for all nonnegative function u that is harmonic in Ω.

Proof. Since $\Omega'\subset\subset\Omega$, $\operatorname{dist}(\bar{\Omega}',\partial\Omega)>0$. Let $r=\frac{1}{4}\operatorname{dist}(\bar{\Omega}',\partial\Omega)$. Now choose $x,y\in\Omega'$, $|x-y|\leqslant r$. Then

$$u(x)=\frac{1}{|B_{2r}|}\int_{B_{2r}(x)}u(z)\,\mathrm{d}z\geqslant\frac{1}{|B_{2r}|}\int_{B_r(y)}u(z)\,\mathrm{d}z=\frac{1}{2^n}\,u(y).^5 \tag{25}$$

4. *That is, the closure $\overline{\Omega'}$ is compact and contained in Ω.*

5. Note that we have used the assumption $u\geqslant 0$ in the inequality.

In other words, for any ball B of radius $r/2$, we have

$$2^n\, u(y) \geqslant u(x) \geqslant \frac{1}{2^n}\, u(y), \qquad \forall x, y \in B \cap \Omega'. \tag{26}$$

Since $\Omega' \subset\subset \Omega$ is connect, $\bar{\Omega}'$ can be covered by a chain of finitely many balls B_i ($i = 1, 2, \ldots, k$), each of radius $r/2$, and $B_i \cap B_{i-1} \neq \phi$. We see that the constant in the theorem is just 2^{nk}. □

4. Poisson representation formula.

Recalling the form of the Green's function for the Laplacian on the ball B_R, we easily obtain the following Poisson representation formula,

$$u(x) \equiv \frac{R^2 - |x|^2}{|B_1|\, R} \int_{\partial B_R} \frac{u(y)}{|x-y|^n}\, \mathrm{d}S, \qquad |x| < R, \tag{27}$$

for any harmonic function u. Therefore

$$u(x) \equiv \frac{R^2 - |x|^2}{|B_1|\, R} \int_{\partial B_R} \frac{\varphi(y)}{|x-y|^n}\, \mathrm{d}S, \qquad |x| < R,$$

should solve the Laplace equation

$$\triangle u = 0 \qquad \text{in } B_R \tag{28}$$

$$u = \varphi \qquad \text{on } \partial B_R. \tag{29}$$

where $\varphi \in C(\partial B_R)$.

We only need to show that u is continuous up to the boundary, that is

$$u(x_n) \to \varphi(x_\infty) \tag{30}$$

for any $x_\infty \in \partial B_R$ and $x_n \to x_\infty$ from inside.

To see this, first note that since $u \equiv 1$ is harmonic, we have

$$\int_{\partial B_R} \frac{R^2 - |x|^2}{|B_1|\, R} \frac{1}{|x-y|^n}\, \mathrm{d}S_y = 1. \tag{31}$$

From now on we denote

$$K(x, y) = \frac{R^2 - |x|^2}{|B_1|\, R} \frac{1}{|x-y|^n} \tag{32}$$

Note that $K \geqslant 0$ and satisfies

$$\lim_{|x'|<1, x' \to x_0} \int_{|y-x_0|>\delta} K(x', y)\, \mathrm{d}y \to 0 \tag{33}$$

for any fixed δ.

All we need to do is to show that

$$\lim_{x \to x_0 \in \partial B_R} \int_{\partial B_R} K(x, y)\, \varphi(y)\, \mathrm{d}y = \varphi(x). \tag{34}$$

Since $\int_{\partial B_R} K(x, y)\, \mathrm{d}S_y = 1$, we have

$$\int_{\partial B_R} K(x, y)\, \varphi(y)\, \mathrm{d}y - \varphi(x_0) = \int_{\partial B_R} K(x, y)\, [\varphi(y) - \varphi(x_0)]\, \mathrm{d}y. \tag{35}$$

For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \frac{\varepsilon}{2} \qquad \text{whenever } |x - x_0| < 2\,\delta, x \in \partial B_R \tag{36}$$

Take $\delta' > 0$ so small that

$$\int_{|y-x_0|>2\delta} K(x', y)\, \mathrm{d}y < \frac{\varepsilon}{2\,M} \qquad \text{whenever } |x' - x_0| < \delta' \tag{37}$$

where $M = \sup_{\partial B_R} |\varphi|$.

We estimate

$$\begin{aligned}
|u(x') - \varphi(x_0)| &= \left| \int_{\partial B_R} K(x', y) \, [\varphi(y) - \varphi(x_0)] \, \mathrm{d}y \right| \\
&\leqslant \left| \int_{|y - x_0| < 2\delta} K(x', y) \, [\varphi(y) - \varphi(x_0)] \, \mathrm{d}y \right| \\
&\quad + \left| \int_{|y - x_0| \geqslant 2\delta} K(x', y) \, [\varphi(y) - \varphi(x_0)] \, \mathrm{d}y \right| \\
&\leqslant \frac{\varepsilon}{2} + 2 \, M \int_{|y - x_0| \geqslant 2\delta} K(x', y) \, \mathrm{d}y < \varepsilon. \qquad (38)
\end{aligned}$$

Note that the above argument gives the existence of the Laplace equation on a ball:

$$\triangle u = 0, \qquad x \in B_R, \qquad u = \varphi, \quad x \in \partial B_R \qquad (39)$$

for $\varphi \in C(\partial B_R)$.

5. Subharmonic and superharmonic functions.

To obtain estimates of the Poisson equation (and more complicated equations)

$$\triangle u = f \qquad (40)$$

it is advantageous to extend maximum principles to the case where $\triangle u$ is not identically 0.

A simple adaptation of the proof for the Laplace equation case gives

Lemma 11. *If $u \in C^2(\Omega)$ satisfies $\triangle u \geqslant 0 (\leqslant 0)$, then*

$$u(x) \leqslant (\geqslant) \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, \mathrm{d}x \text{ and also } \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, \mathrm{d}S. \qquad (41)$$

for every ball $B_r(x) \subset \Omega$.

In light of this lemma, we would like to call $u(x)$ subharmonic when $\triangle u \geqslant 0$ and superharmonic when $\triangle u \leqslant 0$. Note that u is harmonic exactly when u is both subharmonic and superharmonic. And we have

- If u is subharmonic, then $\sup_\Omega u \leqslant \sup_{\partial\Omega} u$, and if $u(x_0) = \sup_{\partial\Omega} u$ for some $x_0 \in \Omega$, then u is constant.
- If u is superharmonic, then $\inf_\Omega u \geqslant \inf_{\partial\Omega} u$, and if $u(x_0) = \inf_{\partial\Omega} u$ for some $x_0 \in \Omega$, then u is constant.

One way to remember these results is to draw a picture of the 1D case. Note that u is subharmonic if and only if $-u$ is superharmonic. Thus in the following we will only discuss subharmonic functions.

However in practice it is usually advantageous to be able to draw similar conclusions for function with less regularity. Motivated by the harmonic case, we can use the following definitions.

Definition 12. (Subharmonic and superharmonic functions)

An upper(lower) semicontinuous[6] function v: $\Omega \mapsto [-\infty, \infty)$, not identically $-\infty$, is subharmonic(superharmonic) if for every ball $B_r(x_0) \subset \Omega$,

$$v(x_0) \leqslant (\geqslant) \frac{1}{|B_r|} \int_{B_r(x_0)} v(x) \, \mathrm{d}x, \qquad (42)$$

it can also be defined by

$$v(x_0) \leqslant (\geqslant) \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v(x) \, \mathrm{d}S. \qquad (43)$$

From this definition the maximum principles immediately follow.

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6. An upper(lower) semicontinuous function is the limit of a monotonically decreasing(increasing) sequence of continuous functions. In particular, every continuous function is upper(lower) semicontinuous.

One can also have the following equivalent characterization, which justifies the usage of the adjectives subharmonic (superharmonic).

Lemma 13. *Let $v: \Omega \mapsto [-\infty, \infty)$ be upper semicontinuous, but not identically $-\infty$. Then v is subharmonic(superharmonic) if and only if for every $\Omega' \subset\subset \Omega$ and every $C^0(\bar{\Omega}')$ harmonic function $u: \Omega' \mapsto \mathbb{R}$ with $v \leqslant (\geqslant) u$ on $\partial\Omega'$, we have*

$$v \leqslant (\geqslant) u \qquad \text{in } \Omega'. \tag{44}$$

Proof. We discuss the two directions.

- "If".

  The basic idea is the following. For any ball $B_r(x_0)$, we can take u harmonic in B_r and $u = v$ on ∂B_r. Then obviously we have

$$v(x_0) \leqslant u(x_0) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} u(x) \, \mathrm{d}S = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v(x) \, \mathrm{d}S. \tag{45}$$

  Integrating w.r.t. r gives

$$v(x_0) \leqslant \frac{1}{|B_r|} \int_{B_r(x_0)} v(x) \, \mathrm{d}x. \tag{46}$$

  This argument doesn't work per se because we do not assume v to be continuous and therefore the existence of the harmonic u is not guaranteed (recall that the Poisson representation formula requires the boundary value to be continuous).

  The way to overcome this technical difficulty is to take a monotonically decreasing sequence of continuous function $v_n \rightarrow v$ where the convergence is pointwise. For each v_n a harmonic u_n exists with $u_n = v_n \geqslant v$ on $\partial B_r(x_0)$. Then we have

$$v(x_0) \leqslant u_n(x_0) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} u_n(x) \, \mathrm{d}S = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v_n(x) \, \mathrm{d}S. \tag{47}$$

  Now since v_n is decreasing, there is a finite uniform upper bound $\max_{\partial B_r(x_0)} v_1(x)$, application of Lebesgue's monotone convergence theorem to $-v_n$ gives

$$\frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v(x) \, \mathrm{d}S = \lim_{n \nearrow \infty} \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} v_n(x) \, \mathrm{d}S. \tag{48}$$

- "Only if". We know that any subharmonic v satisfies the maximum principle:

$$\sup_{\Omega} v \leqslant \sup_{\partial\Omega} v. \tag{49}$$

  Now for any v subharmonic, and u harmonic, $v - u$ is again subharmonic. If $u = v$ on $\partial\Omega'$, $v - u$ has 0 boundary value. This implies $v \leqslant u$ in Ω'. □

Example 14. Examples of subharmonic functions.

1. Let $n \geqslant 2$. We compute

$$\triangle |x|^k = (n\,k + k\,(k-2))\,|x|^{k-2}. \tag{50}$$

   Thus $|x|^k$ is subharmonic when $k \geqslant 2 - n$. Recall that it is harmonic when $k = 2 - n$ (except at 0).

2. Let $u: \Omega \mapsto \mathbb{R}$ be harmonic and positive, $\beta \geqslant 1$, then

$$\triangle(u^\beta) = \sum_1^n \left(\beta\, u^{\beta-1}\, u_{x_i x_i} + \beta\,(\beta-1)\, u^{\beta-2}\, u_{x_i}\, u_{x_i}\right) = \beta\,(\beta-1)\, u^{\beta-2} \sum_1^n u_{x_i}\, u_{x_i}. \tag{51}$$

   Thus u^β is subharmonic if $u \geqslant 0$ and $\beta \geqslant 1$.

3. Let $u: \Omega \mapsto \mathbb{R}$ be harmonic and positive. Then

$$\triangle(\log u) = \sum_1^n \left(\frac{u_{x_i x_i}}{u} - \frac{u_{x_i}\, u_{x_i}}{u^2}\right) = -\sum_1^n \frac{u_{x_i}\, u_{x_i}}{u^2} \leqslant 0. \tag{52}$$

Thus $\log u$ is superharmonic, or equivalently $-\log u$ is subharmonic.

4. Let f be a function which can be approximated locally uniformly by f_n which are C^2 and convex ($f_n'' \geqslant 0$), then $f \circ u$ is subharmonic.

Exercises.

Exercise 1. Prove the $|\alpha| = k$ case of the derivative estimates.

Exercise 2. Prove the following Liouville theorem:

Theorem. (Liouville) *Let $u: \mathbb{R}^d \mapsto \mathbb{R}$ be harmonic and bounded. Then u is constant.*

Hint: Let x, y be any two points. Represent $u(x), u(y)$ using mean value formula over balls of radius R. Compute the difference $u(x) - u(y)$ and then let $R \nearrow \infty$.

Exercise 3. Prove the following Harnack convergence theorem.

Theorem. *Let $u_n: \Omega \mapsto \mathbb{R}$ be a monotonically increasing sequence of harmonic functions. If there exists $y \in \Omega$ for which the sequence $\{u_n(y)\}$ is bounded, then u_n converges on any subdomain $\Omega' \subset\subset \Omega$ uniformly towards a harmonic function.*

Hint: Use Harnack's inequality.