The Laplace/Poisson Equations: Explicit Formulas

In this lecture we study the properties of the Laplace equation

$$\triangle u = 0, \qquad x \in \Omega \subset \mathbb{R}^d \tag{1}$$

and the Poisson equation

$$\triangle u = f, \qquad x \in \Omega \subset \mathbb{R}^d \tag{2}$$

with Dirichlet boundary conditions

$$u = g \qquad x \in \partial\Omega \tag{3}$$

through explicit representations of solutions.

We call a function u harmonic in a region Ω if $\triangle u = 0$ in Ω.

1. Fundamental solution.

Recall that to solve a linear constant-coefficient PDE $P(D)u = f$ in $\mathbb{R}^d$, it suffices to find $\Phi \in \mathcal{D}'$ such that

$$P(D)\Phi = \delta. \tag{4}$$

Now we try to find such Φ for $P(D) = \triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$.

We notice that the equation $\triangle\Phi = \delta$ is invariant with respect to rotations/reflections, in other words, if $\Phi(x)$ is a solution, then $\Phi(O\,x)$ is also a solution where O is an orthogonal matrix. Therefore it is reasonable to suspect that the solution takes the form

$$\Phi(x) = V(r), \qquad r = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}. \tag{5}$$

Simple computation yields

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)^{-1/2} 2\,x_i = \frac{x_i}{r} \qquad x \neq 0, \tag{6}$$

which gives

$$\frac{\partial \Phi}{\partial x_i} = V'(r)\,\frac{x_i}{r}; \qquad \frac{\partial^2\Phi}{\partial x_i^2} = V''(r)\,\frac{x_i^2}{r^2} + V'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right). \tag{7}$$

Summing up we have

$$\triangle\Phi = V''(r) + \frac{n-1}{r}\,V'(r) \tag{8}$$

when $r > 0$.

Since $\triangle\Phi = \delta$, from general theory of distributional solutions of elliptic PDEs we know that

$$\operatorname{sing\,supp} G \subset \operatorname{sing\,supp} \delta = \{0\}. \tag{9}$$

Therefore Φ is C^∞ in $\mathbb{R}^n \backslash \{0\}$, which means $\triangle\Phi = 0$ holds in the classical sense for $x \neq 0$.

Thus it is reasonable to solve

$$\triangle\Phi = 0 \qquad x \neq 0. \tag{10}$$

first to narrow down the candidates. When Φ depends only on $r = |x|$, we have

$$V''(r) + \frac{n-1}{r}\,V'(r) = 0 \qquad r > 0. \tag{11}$$

This gives

$$\log(V')' = \frac{1-n}{r} \quad \Longrightarrow \quad V'(r) = \frac{a}{r^{n-1}} \tag{12}$$

for some constant a. Integrating we obtain

$$V(r) = \begin{cases} b\ln r + c & n = 2 \\ \dfrac{b}{r^{n-2}} + c & n \geqslant 3 \end{cases} \tag{13}$$

for constants b and c.

In particular, we can take special b and c to obtain

Definition 1. *The distribution*

$$\Phi(x) \equiv \begin{cases} \dfrac{1}{2\,\pi} \ln|x| & n = 2 \\ \dfrac{1}{(2 - n)\, n\, \alpha(n)} \dfrac{1}{|x|^{n-2}} & n = 3 \end{cases} \tag{14}$$

where $\alpha(n)$ is the volume of the n-dimensional unit ball (or equivalently, $n\,\alpha(n)$ is the area of the $n-1$-dimensional unit sphere), is called the fundamental solution *of the Laplace's equation.*

One can verify that $\triangle\Phi = \delta$ holds in the sense of distributions. According to the definition of distributional derivatives,

$$(\triangle\Phi)(\phi) \equiv (-1)^2\, \Phi(\triangle\phi) = \Phi(\triangle\phi) = \int_{\mathbb{R}^n} \Phi \triangle\phi, \tag{15}$$

where the last equality comes from the fact that Φ is locally integrable, we only need to show that

$$\int_{\mathbb{R}^n} \Phi \triangle\phi = \phi(0) \tag{16}$$

for any $\phi \in C_0^\infty$. To show this, we need to first establish the so-called Green's formula:

Lemma 2. *Let u, v be C^2 in Ω, then*

$$\int_\Omega u \triangle v - v \triangle u \,\mathrm{d}x = \int_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \,\mathrm{d}S. \tag{17}$$

where n is the outward unit normal vector of $\partial\Omega$.

Proof. Recall the Gauss theorem:

$$\int_\Omega \nabla \cdot \boldsymbol{f} \,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n} \,\mathrm{d}S. \tag{18}$$

Now we have

$$\begin{aligned} \int u \triangle v - v \triangle u \,\mathrm{d}x &= \int [\nabla \cdot (u \nabla v) - \nabla u \cdot \nabla v] - [\nabla \cdot (v \nabla u) - \nabla v \cdot \nabla u] \,\mathrm{d}x \\ &= \int \nabla \cdot (u \nabla v - v \nabla u) \,\mathrm{d}x \\ &= \int_{\partial\Omega} u\, (\boldsymbol{n} \cdot \nabla v) - v\, (\boldsymbol{n} \cdot \nabla u) \,\mathrm{d}S \\ &= \int_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \,\mathrm{d}S. \end{aligned} \tag{19}$$

□

Now we show that $\triangle\Phi = \delta$ for $n = 2$, and leave the $n \geqslant 3$ case as an exercise.

Take any $\phi \in C_0^\infty(\mathbb{R}^2)$, there is $R > 0$ such that $\operatorname{supp} \phi \subset B_R$ where B_R denotes the open ball centered at the origin and with radius R. Now take $\varepsilon > 0$ small. We set $\Omega = B_R \backslash B_\varepsilon$ and apply the Green's formula to $u = \Phi$ and $v = \phi$. Keep in mind that $\partial\Omega$ has two parts.

Since $\triangle\Phi = 0$ in Ω, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \Phi \triangle\phi &= \lim_{\varepsilon \searrow 0} \int_{B_R \backslash B_\varepsilon} \Phi \triangle\phi - \phi \triangle\Phi \\ &= \lim_{\varepsilon \searrow 0} \int_{\partial B_R} \Phi \frac{\partial\phi}{\partial n_{\text{out}}} - \phi \frac{\partial\Phi}{\partial n_{\text{out}}} + \lim_{\varepsilon \searrow 0} \int_{\partial B_\varepsilon} \Phi \frac{\partial\phi}{\partial n_{\text{in}}} - \phi \frac{\partial\Phi}{\partial n_{\text{in}}} \\ &\equiv A + B + C + D. \end{aligned} \tag{20}$$

Here $n_{\text{out}} = \frac{x}{R}$ for $x \in \partial B_R$ while $n_{\text{in}}(x) = -\frac{x}{\varepsilon}$ for $x \in \partial B_\varepsilon$.

Checking term by term, we have

- $A = 0$ since $\operatorname{supp} \phi \subset B_R$ which means $\frac{\partial\phi}{\partial n_{\text{out}}} = 0$.
- $B = 0$ for the same reason.

– For C, we have

$$\left| \int_{\partial B_\varepsilon} \Phi \frac{\partial \phi}{\partial n_{\text{in}}} \right| \leqslant \int_{\partial B_\varepsilon} |\log \varepsilon| \sup_{x \in \partial B_\varepsilon} |\nabla \phi| \leqslant C \varepsilon |\log \varepsilon|, \tag{21}$$

therefore $C = 0$.

– Finally, for D, we have for $x \in \partial B_\varepsilon$,

$$\frac{\partial \Phi}{\partial n_{\text{in}}} = -\frac{x}{\varepsilon} \cdot \nabla \Phi = -\frac{x}{\varepsilon} \cdot \left(\frac{1}{2\pi} \right) \left[\frac{1}{r} \frac{x}{r} \right] |_{r = |x| = \varepsilon} = -\frac{1}{2\pi\varepsilon}. \tag{22}$$

Thus

$$D = \lim - \int \phi \frac{\partial \Phi}{\partial n_{\text{in}}} = \lim \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon} \phi = \phi(0). \tag{23}$$

In summary, $u = \Phi * f$ solves the Poisson equation

$$\triangle u = f \tag{24}$$

in the sense of distributions for any $f \in \mathcal{E}'(\mathbb{R}^n)$.

2. The Green's function for the Laplace equation.

In practice, we are more interested in solving the Poisson equation on a domain with boundary instead of the full space.

We explicitly define

$$\Gamma(a, x) = \Phi(x - a) \tag{25}$$

where Φ is the fundamental solution, to make the presentation clearer.

The Dirichlet problem takes the form

$$\triangle u = f \qquad \text{in } \Omega, \qquad u = g \qquad \text{on } \partial\Omega \tag{26}$$

In this case, we have

Theorem 3. *Suppose Ω is a bounded domain in $\mathbb{R}^n$, and that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then for any $a \in \Omega$ there holds*

$$u(a) = \int_\Omega \Gamma(a, x) \triangle u(x) \,\mathrm{d}x - \int_{\partial\Omega} \left(\Gamma(a, x) \frac{\partial u}{\partial n_x}(x) - u(x) \frac{\partial \Gamma}{\partial n_x}(a, x) \right) \mathrm{d}S_x. \tag{27}$$

The proof uses the Green's formula and can be found in F. Lin and Q. Han **Elliptic Partial Differential Equations**. One further notices that for any $\Psi(a, x)$ satisfying $\triangle_x \Psi = 0$ (here $\triangle_x$ denotes the Laplacian with respect to the variable x), we have

$$u(a) = \int_\Omega \tilde{\Gamma}(a, x) \triangle u(x) \,\mathrm{d}x - \int_{\partial\Omega} \left(\tilde{\Gamma}(a, x) \frac{\partial u}{\partial n_x}(x) - u(x) \frac{\partial \tilde{\Gamma}}{\partial n_x}(a, x) \right) \mathrm{d}S_x. \tag{28}$$

where

$$\tilde{\Gamma}(a, x) = \Gamma(a, x) + \Psi(a, x). \tag{29}$$

Using the equation and boundary conditions, we have

$$u(a) = \int_\Omega \tilde{\Gamma}(a, x) f(x) \,\mathrm{d}x + \int_{\partial\Omega} g(x) \frac{\partial \tilde{\Gamma}}{\partial n_x}(a, x) \,\mathrm{d}S - \int_{\partial\Omega} \tilde{\Gamma}(a, x) \frac{\partial u}{\partial n_x}(x). \tag{30}$$

To be able to find $u(a)$, all we need to do is to find a function $G(a, x)$ such that

1. $G(a, x) = \Gamma(a, x) + \Psi(a, x)$ where Ψ is harmonic w.r.t. x,
2. $G(a, x) = 0$ for $x \in \partial\Omega$.

For such G, we have

$$u(a) = \int_\Omega G(a, x) f(x) \,\mathrm{d}x + \int_{\partial\Omega} g(x) \frac{\partial G}{\partial n_x}(a, x) \,\mathrm{d}S. \tag{31}$$

where both terms on the RHS are known (of course only when we can construct this G!).

The conditions on G implies that Ψ solves

$$\triangle \Psi = 0 \qquad \text{in } \Omega, \qquad \Psi(a,x) = -\Gamma(a,x) \qquad \text{on } \partial\Omega. \tag{32}$$

This can be done by cleverly transforming and combining Γ's when the domain has simple geometry. However when for general domains obviously the problem is as hard as the original Dirichlet problem for u.

Example 4. (Green's function for the ball B_R) For each $a \in B_R$, we need to find $\Psi(a,x)$ such that

$$\triangle_x \Psi(a,x) = 0 \qquad \text{in } B_R; \qquad \Psi(a,x) = -\Gamma(a,x) \qquad \text{on } \partial B_R. \tag{33}$$

For simplicity of presentation, we abuse notation and write $\Gamma(a,x)$ as $\Gamma(|x-a|)$.

The idea is to find a point $b = b(a) \notin B_R$ and a constant c, then set $\Psi(a,x) = -\Gamma(c\,|x - b(a)|)$. Note that for any $b \notin B_R$, $-\Gamma(c\,|x-b|)$ is harmonic in B_R. Thus all we need to do is finding appropriate c and b so that

$$\Gamma(|x-a|) = \Gamma(c\,|x-b|). \tag{34}$$

From the explicit formula of Γ we see that this can hold if and only if $|x-a| = c\,|x-b|$ for all $x \in \partial B_R$. Let $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. Then $|x-a| = c\,|x-b|$ becomes

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 = c^2 \left[(x_1 - b_1)^2 + \cdots + (x_n - b_n)^2 \right] \tag{35}$$

Simplifying, we obtain

$$(1 - c^2)\left[x_1^2 + \cdots + x_n^2\right] = 2\left[(a_1 - c^2 b_1)x_1 + \cdots + (a_n - c^2 b_n)\,x_n\right] + c^2 \sum b_i^2 - \sum a_i^2. \tag{36}$$

For this to hold for all $x \in \partial B_R$, we need

$$(1 - c^2)\,R^2 = 2\left[(a_1 - c^2 b_1)x_1 + \cdots + (a_n - c^2 b_n)\,x_n\right] + c^2 \sum b_i^2 - \sum a_i^2 \tag{37}$$

for all $x \in B_R$. As a consequence we have

$$a_i = c^2 b_i \qquad i = 1, \ldots, n \tag{38}$$

$$(1 - c^2)\,R^2 = c^2 \sum b_i^2 - \sum a_i^2 \tag{39}$$

whose solutions are

$$c^2 = \frac{\sum a_i^2}{R^2} = \frac{|a|^2}{R^2} \tag{40}$$

$$b = a/c^2 = \frac{R^2}{|a|^2}\,a. \tag{41}$$

Thus

$$\Psi(a,x) = -\Gamma\left(\frac{|a|}{R} \left| x - \frac{R^2}{|a|^2}\,a \right| \right) \tag{42}$$

and

$$G(a,x) = \Gamma(|x-a|) - \Gamma\left(\frac{|a|}{R} \left| x - \frac{R^2}{|a|^2}\,a \right| \right). \tag{43}$$

Remark 5. One easily checks that $G(a,x) = G(x,a)$ for the Green's function on B_R, by writing

$$\frac{|a|}{R} \left| x - \frac{R^2}{|a|^2}\,a \right| = \left| \frac{|a|}{R}\,x - \frac{R}{|a|}\,a \right| = \left(\frac{|a|^2\,|x|^2}{R^2} - 2\,a \cdot x + R^2 \right)^{1/2}. \tag{44}$$

This turns out to remain true for other Green's functions. A formal argument to convince oneself about this is the following.

Let $a \in \Omega$ be arbitrary. We would like to show $G(a, x) = G(x, a)$ for all $x \in \Omega$. Writing $G_1(x) \equiv G(a, x)$, the goal becomes to show that $G_1(x) = G(x, a)$.

Now $G_1(x)$ solves

$$\triangle G_1 = \delta_a, \qquad G_1 = 0 \quad \text{on } \partial\Omega, \tag{45}$$

Thus formally we have

$$G_1(x) = \int_\Omega G(x, y)\, \delta_a(y) = G(x, a). \tag{46}$$

Obviously the above argument has several "holes". For example $G(x,\, y) \notin C(\Omega)$ so the action of $\delta_a(y)$ on it is not defined. However one can make it rigorous by computing

$$\int_{\Omega \backslash B_\varepsilon(x_1) \cup B_\varepsilon(x_2)} (G_1 \triangle G_2 - G_2 \triangle G_1)\, \mathrm{d}x \tag{47}$$

where $G_1(x) = G(x_1, x)$ and $G_2(x) = G(x, x_2)$, and then let $\varepsilon \searrow 0$. See L. C. Evans **Partial Differential Equations** for details.

3. Harmonic functions.

A function $u \in C^2(\Omega)$ is said to be harmonic if $\triangle u = 0$. Let g be its boundary value, we see that

$$u(a) = \int_{\partial\Omega} g(x)\, \frac{\partial G}{\partial n_x}(a, x)\, \mathrm{d}S_x. \tag{48}$$

In particular, when $\Omega = B_R$, we have the Poisson representation formula

$$u(a) = \frac{R^2 - |a|^2}{n\, \alpha(n)\, R} \int_{\partial B_R} \frac{g(x)}{|a - x|^n}\, \mathrm{d}S_x. \tag{49}$$

4. Well-posedness.

Given the Dirichlet problem

$$\triangle u = f, \qquad u = g \quad \text{on } \partial\Omega, \tag{50}$$

as soon as we have the Green's function, we can write

$$u(a) = \int_\Omega G(a, x)\; f(x)\, \mathrm{d}x + \int_{\partial\Omega} g(x)\, \frac{\partial G}{\partial n_x}(a, x)\, \mathrm{d}S_x. \tag{51}$$

There are several issues need to be settled (they will be settled in the following few weeks).

- Existence.

  The formula itself does not give us existence of the solution per se. We need to show that $\triangle u = f$ indeed holds and u takes g as its boundary value. In light of the next section, we can set $v = u - \int \Gamma(x - y)\, f(y)$ which satisfies the Laplace equation $\triangle v = 0$ with some boundary conditions. Thus the question becomes whether we can show the existence of solutions to the Laplace equation.

  When $\Omega = B_R$, it suffices to show that the Poisson integral representation indeed gives a solution to the Laplace equation.

  For general Ω, it is not possible to find a Green's function with explicit formula. It turns out somehow one can show the existence of solution to the Laplace equation $\triangle u = 0$ through solving it iteratively on balls inside the domain. This is the Perron's method.

- Uniqueness.

  It can be easily seen that if u_1, u_2 solves the same Poisson's equation, their difference $u_1 - u_2$ satisfies the Laplace equation with zero boundary condition. Thus we only need to show that the zero function is the only solution in this situation. This is a consequence of the maximum principle.

- Regularity.

  In the following section we prove a trivial regularity result. In the next lecture we will prove much more refined regularity versions.

5. Regularity – a glimpse.

Let $\Phi(x)$ be the fundamental solution of the Laplacian. Assuming $f \in C_c^2(\mathbb{R}^n)$ (twice continuously differentiable with compact support), then we have $u = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y) \in C^2(\mathbb{R}^n)$.

First write

$$u = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y) = \int_{\mathbb{R}^n} \Phi(y)\, f(x-y). \tag{52}$$

Recall that Φ is locally integrable, thus by assumption both $\Phi(y)\, \frac{\partial}{\partial x_i} f(x-y)$ and $\Phi(y)\, \frac{\partial^2}{\partial x_i \partial x_j} f(x-y)$ are integrable. As a consequence one has

$$\frac{\partial u}{\partial x_i} = \int \Phi(y)\, \frac{\partial}{\partial x_i} f(x-y) \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i\, \partial x_j} = \Phi(y)\, \frac{\partial^2}{\partial x_i \partial x_j} f(x-y) \tag{53}$$

following from properties of the Lebesgue integrals.[1] Since both $\frac{\partial}{\partial x_i} f(x-y)$ and $\frac{\partial^2}{\partial x_i \partial x_j} f(x-y)$ are uniformly continuous, $u \in C^2$.

The above result is obviously not satisfactory as no improvement on regularity is obtained, while intuitively, since f is a linear combination of the double derivatives of u, one would expect the regularity of u to be better than that of f. For example, it is natural to guess that $f \in C \Longrightarrow u \in C^2$ (which is obviously true in the one dimensional case). Unfortunately this particular conjecture is wrong, but u turns out to be indeed twice more differentiable than f if we use the "right" function spaces. We will discuss this issue more in the next lecture.

Further readings.

- L. C. Evans, **Partial Differential Equations**, §2.2.
- D. Gilbarg, N. S. Trudinger, **Elliptic Partial Differential Equations of Second Order**, Chapter 2, Classics in Mathematics, Springer.
- Fanghua Lin, Qing Han, **Elliptic Partial Differential Equations**, §1.3, Courant Lecture Notes 1, AMS.

Exercises.

Exercise 1. Prove that the fundamental solution for dimensional at least 3:

$$\Phi(x) = \frac{1}{(2-n)\, n\, \alpha(n)} \frac{1}{|x|^{n-2}} \tag{55}$$

solves the equation

$$\triangle \Phi = \delta \tag{56}$$

in the sense of distributions. Remember that $n\, \alpha(n)$ is the area of the unit sphere in $\mathbb{R}^n$. results.

Exercise 2. Construct the Green's function for the half-space $\{x_n > 0\}$.

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1. One can also prove directly using finite differences. For example, let e_i be the unit vector in x_i direction, that is $e_i = (0, \ldots, 1, \ldots, 0)$ with the only 1 in the i-th position. Then

$$\frac{u(x + \varepsilon\, e_i) - u(x)}{\varepsilon} = \int_{\mathbb{R}^n} \Phi(y)\, \frac{f(x + \varepsilon\, e_i - y) - f(x-y)}{h}\, \mathrm{d}y. \tag{54}$$

Since $f \in C_c^2$, $\frac{f(x + \varepsilon\, e_i - y) - f(x-y)}{h} \longrightarrow \frac{\partial f}{\partial x_i}(x-y)$ uniformly because $\frac{\partial f}{\partial x_i}$ is uniformly continuous.