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Distributions

Distributions are generalized functions. Some familiarity with the theory of distributions helps understanding of various function spaces which play important roles in the study of PDEs. In this lecture we will introduce the theory of distributions, and illustrate its power show via examples.

1. Distributions.

1.1. Introduction.

Definition 1. (Distributions) *A distribution T on Ω is a linear functional on $C_0^\infty(\Omega)$ such that $T(\phi_j) \to 0$ for every sequence $\{\phi_j\} \subset C_0^\infty(\Omega)$ converging to 0, that is satisfying*[1]

i. the supports of all ϕ_js are in a common compact set $K \subset \Omega$. Here K is independent of j.

ii. $\sup_{x \in K} |D^\alpha \phi_j| \to 0$ as $j \to \infty$ for all α with $|\alpha| \geqslant 0$.

We denote the space of distributions on Ω by $\mathcal{D}'(\Omega)$[2]*.*

Example 2. The following linear functionals are distributions.

1. Let f be a locally integrable function on $\mathbb{R}^d$. Define linear functional T_f by
$$T_f(\phi) = \int_{\mathbb{R}^n} f \phi. \tag{2}$$
2. Let $\boldsymbol{a} \in \mathbb{R}^d$. Consider the Dirac delta function:
$$\delta_{\boldsymbol{a}}(\phi) \equiv \phi(\boldsymbol{a}). \tag{3}$$
3. Let $\boldsymbol{a} \in \mathbb{R}^d$, and α a multi-index. Define
$$\delta_a^\alpha(\phi) = (-1)^{|\alpha|} (D^\alpha \phi)(\boldsymbol{a}). \tag{4}$$

- If $\{T_j\} \subset \mathcal{D}'(\Omega)$ satisfies $T_j \phi \to 0$ for any $\phi \in C_0^\infty(\Omega)$, we say T_j converges to 0 in $\mathcal{D}'(\Omega)$, and write $T_j \to 0$. We say $T_j \to T$ if $T_j - T \to 0$. It turns out that $\mathcal{D}'(\Omega)$ is complete under this topology[3].
- For $T \in \mathcal{D}'(\Omega)$, we say that T is zero on an open set $V \subseteq \Omega$ if $T(\phi) = 0$ for every $\phi \in C_0^\infty(V)$. The support of $T \in \mathcal{D}'(\Omega)$ is defined as follows:
$$\operatorname{supp} T = \{x \in \Omega \mid T \not\equiv 0 \text{ on any neighborhood of } x\}. \tag{6}$$

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1. This topology is "induced" from the natural topology on the space C^∞ defined by the semi-norms
$$p_{R,k}(u) \equiv \sum_{|\alpha| \leqslant k} \sup_{|x| \leqslant R} |D^\alpha u|. \tag{1}$$

2. The reason for such notation is that L. Schwartz used $\mathcal{D}$ to denote C_0^∞ in his original paper.

3. We need to show that if $T_n(\phi)$ converges for every $\phi \in C_0^\infty$,the limit T defined by
$$T(\phi) = \lim_{n \nearrow \infty} T_n(\phi) \tag{5}$$
is also a distribution.

This is a direct consequence of the Banach-Steinhaus Theorem in functional analysis (note that C_0^∞ is metrizable). But one can also prove it directly via the following argument.

1. If $T \notin \mathcal{D}'$, then there are $\phi_n \to 0$ but $|T(\phi_n)| \geqslant c > 0$. Replacing ϕ_n by $c_n \phi_n$ for appropriate c_n we can make $|T(\phi_n)| \nearrow \infty$. We want to construct $\psi \in C_0^\infty$ such that $T_n(\psi) \not\to 0$, thus getting a contradiction. The idea is to find subsequences ψ_i and S_i, such that $S_i(\psi_i)$ is large (possible because $T(\psi_i)$ is large, and $S_j(\psi_i)$ small for all $j \neq i$. Then set $\psi = \sum \psi_i$.
2. Now take ϕ_i such that $|T(\phi_i)| > 1$ and then we can find T_j such that $|T_j(\phi_i)| > 1$. Set $\psi_1 = \phi_i$, $S_1 = T_j$.
3. Choose ψ_n such that $|S_i(\psi_n)| < 1/2^{n-j}$ for $j = 1, \ldots, n-1$, and $|T(\psi_n)| > \sum_{j=1}^{n-1} |T(\psi_j)| + n$. Then choose S_n satisfying $|S_n(\psi_n)| > \sum_{j=1}^{n-1} |T(\psi_j)| + n$.
4. Set $\psi = \sum_1^\infty \psi_n$. We have $|S_n(\psi)| > n - 1$.

One can further show that if $T_n \to 0$ in $\mathcal{D}'$ and $\phi_n \to 0$ in C_0^∞, then $T_n(\phi_n) \to 0$.

Example 3. The support of $\delta_{\boldsymbol{a}}^{\alpha}$ is $\{\boldsymbol{a}\}$.

There is an alternative definition which is much more useful in practice. We present it as a lemma.

Lemma 4. *Let $T \in \mathcal{D}'(\Omega)$. Then for any compact subset K of Ω, there is $n = n(K) \in \mathbb{N}$ and $C = C(K)$ such that*

$$|T(\phi)| \leqslant C \sum_{|\alpha| \leqslant n} \max_{x \in K} |D^{\alpha}\phi(x)| \tag{7}$$

for all ϕ supported in K.

Conversely, if the above is satisfied for a linear functional T on C_0^{∞}, then $T \in \mathcal{D}'(\Omega)$.

Proof. The "converse" direction is trivial.

For the other direction, we prove by contradiction. Assume that there is $T \in \mathcal{D}'(\Omega)$ such that there is a compact subset K of Ω, such that there are $\phi_n \in C_0^{\infty}(\Omega)$ supported in K and

$$|T(\phi_n)| > n \sum_{|\alpha| \leqslant n} \max_{x \in K} |D^{\alpha}\phi_n(x)|. \tag{8}$$

By replacing ϕ_n by $c_n \phi_n$, we can set $|T(\phi_n)| = 1$. Thus we have

$$\sum_{|\alpha| \leqslant n} \max_{x \in K} |D^{\alpha}\phi_n(x)| < \frac{1}{n}. \tag{9}$$

But this implies $\phi_n \to 0$, consequently $T(\phi_n) \to 0$ and we obtain a contradiction. □

Remark 5. The constant $n(K)$ may not be uniform. For example, let $T \in \mathcal{D}'(\mathbb{R})$ be defined by

$$T(\phi) = \sum_{n=1}^{\infty} \phi^{(n)}(n). \tag{10}$$

If n can be taken to be independent of K, then the smallest uniform n is called the *order* of the distribution T. For example, the Dirac δ function is a distribution of order 0, the distribution $\delta_{\boldsymbol{a}}^{\alpha}$ has order $|\alpha|$.

Remark 6. It turns out that C_0^{∞} is dense in $\mathcal{D}'$, that is, any distribution T is the limit of a sequence of C_0^{∞} functions f_n, in the sense that for any $\phi \in C_0^{\infty}$,

$$T(\phi) = \lim_{n \nearrow \infty} \int_{\mathbb{R}^d} f_n(x)\, \phi(x)\, \mathrm{d}x. \tag{11}$$

1.2. Operations on distributions.

Differentiation.

Definition 7. (Derivatives of a distribution) *Let $T \in \mathcal{D}'(\Omega)$, and let α be any multi-index of nonnegative integers. The derivative $D^{\alpha}T$ is defined by*

$$D^{\alpha}T(\phi) = (-1)^{|\alpha|}\, T(D^{\alpha}\phi). \tag{12}$$

Example 8. Consider the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{13}$$

H is locally integrable, and we identify it with the distribution T_H. We can show that $H' = \delta$.

If we define K by

$$K(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}. \tag{14}$$

then $K' = H$.

Example 9. Let

$$x_+^\lambda \equiv \begin{cases} 0 & x \leqslant 0 \\ x^\lambda & x > 0 \end{cases} \tag{15}$$

with $-1 < \lambda < 0$.

Differentiating naively, we would have $\left(x_+^\lambda\right)' = \begin{cases} 0 \\ \lambda\, x^{\lambda - 1} \end{cases}$ but it is not locally integrable and is not a distribution. The correct computation is as follows.

$$\begin{aligned} \left(x_+^\lambda\right)'(\phi) &= -\left(x_+^\lambda\right)(\phi') \\ &= -\int_0^\infty x^\lambda\, \phi'(x)\, \mathrm{d}x \\ &= -\lim_{\varepsilon \to 0} \int_\varepsilon^\infty x^\lambda\, \phi'(x)\, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \lambda\, x^{\lambda-1}\, \phi(x)\, \mathrm{d}x + \varepsilon^\lambda\, \phi(\varepsilon) \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \lambda\, x^{\lambda-1}\, \phi(x)\, \mathrm{d}x + \varepsilon^\lambda\, \phi(0) + \varepsilon^\lambda\, [\phi(\varepsilon) - \phi(0)] \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \lambda\, x^{\lambda-1}\, [\phi(x) - \phi(0)]. \end{aligned} \tag{16}$$

Therefore the distributional derivative of x_+^λ is defined by

$$\left(x_+^\lambda\right)'(\phi) = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \lambda\, x^{\lambda-1}\, [\phi(x) - \phi(0)]. \tag{17}$$

Example 10. (Poisson summation formula) We consider the sum

$$\cos x + \cos 2\, x + \cdots. \tag{18}$$

It turns out the limit of this diverging sequence exists as a distribution.

To see this, let $T_n = \sum_1^n \cos k\, x$ and $S_n = \sum_1^n \frac{1}{k} \sin k\, x$. From basic Fourier analysis we know that $S_n \to \frac{\pi - x}{2}$ locally uniformly on $(0, 2\,\pi)$. Since S_n is uniformly bounded, one can show that S_n converges to the periodic expansion of $\frac{\pi - x}{2}$ in $\mathcal{D}'(\mathbb{R})$. On the other hand we see that $T_n = S_n'$ as distributions, thus $T_n \to S'$ where $S = \lim S_n$. Therefore we obtain the summation formula

$$\sum_1^\infty \cos k\, x = -\frac{1}{2} + \pi \sum_{n \in \mathbb{Z}} \delta(x - 2\, n\, \pi). \tag{19}$$

Example 11. Let $T \in \mathcal{D}'(\mathbb{R})$. If $T' = 0$ then T is a constant. To see this, recall that $T' = 0$ implies

$$T(\phi') = 0 \tag{20}$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Now take an arbitrary function $h \in C_0^\infty(\mathbb{R})$ with $\int h = 1$. For any $\psi \in C_0^\infty(\mathbb{R})$ we have

$$\psi(x) = h(x) \int_{\mathbb{R}} \psi(s)\, \mathrm{d}s + \psi_1(x) \tag{21}$$

Since $\int \psi_1(x) = 0$, its primitive $\phi(x) = \int_{-\infty}^x \psi_1(s)\, \mathrm{d}s \in C_0^\infty(\mathbb{R})$ and consequently

$$T(\psi) = T\left(h(x) \int_{\mathbb{R}} \psi(s)\, \mathrm{d}s\right) + T(\psi_1) = (T(h(x))) \int_{\mathbb{R}} \psi(s)\, \mathrm{d}s. \tag{22}$$

In other words,

$$T = T(h(x)) \tag{23}$$

is a constant as a distribution. One can show using similar ideas that any $T \in \mathcal{D}'\left(\mathbb{R}^d\right)$ with $\nabla T = 0$ is a constant.

Multiplication by smooth functions.

Let $T\in\mathcal{D}'(\Omega)$ and $f\in C^\infty(\Omega)$. Then we can define a linear functional fT by

$$(fT)(\phi)=T(f\phi). \tag{24}$$

We see that $fT\in\mathcal{D}'(\Omega)$ by noticing that

1. $f\phi\in C_0^\infty(\Omega)$ when $\phi\in C_0^\infty(\Omega)$;
2. When $\phi_j\to 0$ in $C_0^\infty(\Omega)$, $f\phi_j\to 0$ too.

Convolution.

In general, the product TS for $T,S\in\mathcal{D}'(\Omega)$ is not defined. What we can do is calculate their convolution.

Definition 12. (Convolution of functions) *Let f,g be integrable. We define a new function $f*g$ by*

$$(f*g)(x)=\int_{\mathbb{R}^d} f(y)\,g(x-y)\,\mathrm{d}y. \tag{25}$$

The convolution operation has the following properties:

1. $f*g$ is integrable;
2. $f*g=g*f$; $f*(g*h)=(f*g)*h$;
3. $\operatorname{supp}(f*g)\subseteq\operatorname{supp}f+\operatorname{supp}g$, where the sum of two sets A,B is defined as $A+B=\{x\mid x=y+z, y\in A, z\in B\}$.
4. If f is differentiable, so is $f*g$ (even if g is not), and $D^\alpha(f*g)=(D^\alpha f)*g$.

We can extend the convolution operation to distributions, using the following calculations as a guide:

$$\begin{aligned}\int (f*g)(x)\,\phi(x)\,\mathrm{d}x &= \int\left(\int f(y)\,g(x-y)\,\mathrm{d}y\right)\phi(x)\,\mathrm{d}x\\ &= \int\int f(y)\,g(x-y)\,\phi(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int f(y)\left(\int g(x-y)\,\phi(x)\,\mathrm{d}x\right)\mathrm{d}y\\ &= \int f(y)\left(\int g(z)\,\phi(z+y)\,\mathrm{d}z\right)\mathrm{d}y\\ &= \int f(y)\,(T_g(\phi(\,\cdot+y)))\,\mathrm{d}y\\ &= T_f(T_g(\phi(\,\cdot+y))).\end{aligned} \tag{26}$$

Now let $T,S\in\mathcal{D}'(\Omega)$ and assume S has compact support (denoted as $S\in\mathcal{E}'(\Omega)$[4]), we define

$$(T*S)(\phi)\equiv T_y(S_x(\phi(x+y))). \tag{27}$$

where the subscripts y, x means the distributions T and S are acting in the y-space and x-space respectively.

We can show that the above properties still hold:

1. $(T*S)\in\mathcal{D}'(\Omega)$[5];
2. $T*S=S*T$; If $U\in\mathcal{E}'(\Omega)$, $(T*S)*U=T*(S*U)$;
3. $\operatorname{supp}(T*S)\subseteq\operatorname{supp}T+\operatorname{supp}S$;
4. $D^\alpha(T*S)=(D^\alpha T)*S=T*(D^\alpha S)$. Recall that distributions are infinitely differentiable in the space of distributions.

4. $\mathcal{E}(\Omega)$ is the notation for $C^\infty(\Omega)$ used by Schwartz, and compacted supported distributions form the dual space of it, thus the notation $\mathcal{E}'(\Omega)$.

5. This is not true without the condition on the support of S. More specifically, $S_y(\phi(x+y))$ may not still be in C_0^∞ when the support of S is not compact. Consider for example the distribution S_1 obtained by the constant function 1.

Example 13. Let $T\in\mathcal{D}'(\Omega)$, we compute $T*\delta$. Let $\phi\in C_0^\infty$. We have

$$(T*\delta)(\phi)=T_y(\delta_x(\phi(\cdot+y))=T(\phi). \tag{28}$$

Therefore

$$T*\delta=T \tag{29}$$

More generally,

$$T*(D^\alpha\delta)=D^\alpha T. \tag{30}$$

for any multi-index α.

Example 14. (Fundamental solutions) The power of distributions in the study of linear PDEs comes from the following fact:

Let $P(D)$ be a differential operator with constant coefficients. Let G solves

$$P(D)G=\delta \tag{31}$$

Such a $G\in\mathcal{D}'$ is called a fundamental solution of the operator P.

Then for any $f\in\mathcal{D}'$ we have

$$P(D)(G*f)=[P(D)G]*f=\delta*f=f. \tag{32}$$

In other words $G*f$ solves the equation

$$P(D)u=f. \tag{33}$$

Therefore the solvability of any constant-coefficient equation $P(D)u=f$ reduces to the existence of the fundamental solution G.

Fourier transform of distributions.

In this part we consider the case $\Omega=\mathbb{R}^d$. A very powerful tool in the study of PDEs is the Fourier transform, which turns differential equations into algebraic equations. Unfortunately not all distributions have Fourier transforms. The appropriate subset that has Fourier transforms is called tempered distributions and denoted $\mathcal{S}'$. To do this we need to first define the space of rapidly decreasing functions.

Definition 15. (Rapidly decreasing functions) *We first define the space $\mathcal{S}$ of rapidly decreasing functions:*

$$\mathcal{S}=\left\{f\in C^\infty\big(\mathbb{R}^d\big)\colon \lim_{|x|\to\infty}\big|x^\alpha D^\beta f(x)\big|=0 \quad \textit{for all multi-indices}\,\alpha\ \textit{and}\,\beta\right\}. \tag{34}$$

A sequence $\{\phi_n\}\subset\mathcal{S}$ is said to converge to 0 in $\mathcal{S}$, denoted $\phi_n\to 0$ in $\mathcal{S}$, if

$$\lim_{n\nearrow\infty}\sup_{x\in\mathbb{R}^d}\left|\langle x\rangle^k D^\beta\phi_n(x)\right|=0. \tag{35}$$

for all k,β. Here

$$\langle x\rangle\equiv\big(1+x_1^2+\cdots+x_d^2\big)^{1/2}. \tag{36}$$

Example 16. $e^{-|x|^2}\in\mathcal{S}$. Note that it does not belong to C_0^∞.

One can show that $C_0^\infty\subset\mathcal{S}$ and $\phi_n\to 0$ in C_0^∞ implies $\phi_n\to 0$ in $\mathcal{S}$.

Definition 17. (Tempered distributions) *The space of tempered distributions, denoted $\mathcal{S}'$, consists of those functionals on $\mathcal{S}$ such that if $\phi_n\to 0$ in $\mathcal{S}$, then $T(\phi_n)\to 0$. Equivalently, $T\in\mathcal{S}'$ if there are m,C such that*

$$|T(\phi)|\leqslant C\sum_{|\alpha|\leqslant m}\sup_{x\in\mathbb{R}^d}\big|\langle x\rangle^m D^\alpha\phi(x)\big| \qquad \forall\phi\in\mathcal{S}. \tag{37}$$

Example 18. $\delta_{\boldsymbol{a}}^\alpha\in\mathcal{S}'$ for all point $\boldsymbol{a}$ and multi-index α.

Now we can define the Fourier transform of $T\in\mathcal{S}'$.

Definition 19. **(Fourier transform of tempered distributions)** *Let $T \in \mathcal{S}'$. Its Fourier transform $\hat{T}$ is defined by*

$$\hat{T}(\phi) = T\left(\hat{\phi}\right) \tag{38}$$

for any $\phi \in S$.

Remark 20. The above definition is motivated by the following calculation for $f, g \in S$. [6]

$$T_{\hat{f}}(g) = \int \hat{f}(\xi)\, g(\xi) = \int \left(\int e^{-i\xi\cdot x} f(x) \right) g(\xi) = \int \left(\int e^{-i\xi\cdot x} g(\xi) \right) f(x) = \int \hat{g}(x)\, f(x) = T_f(\hat{g}). \tag{41}$$

Remark 21. For this definition to make sense, it must hold that whenever $\phi \in \mathcal{S}$, $\hat{\phi} \in \mathcal{S}$ too. It turns out that $\hat{\ }$ is an isomorphism on $\mathcal{S}$ (Interested readers can try to verify this). As a consequence, it is also an isomorphism on the dual space $\mathcal{S}'$.

We notice that when $\phi_n \to 0$ in $\mathcal{S}$, so is $\hat{\phi}_n$. Therefore $\hat{T} \in \mathcal{S}'$.

Example 22. We compute $\hat{\delta}$. Take any $\phi \in \mathcal{S}$. we have

$$\hat{\delta}(\phi) = \delta\left(\hat{\phi}\right) = \hat{\phi}(0) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \phi(x)\, \mathrm{d}x. \tag{42}$$

Therefore

$$\hat{\delta} = (2\pi)^{-d/2}\, \mathbf{1} \tag{43}$$

where $\mathbf{1}$ is the constant function.

Example 23. Consider the tempered distribution defined by the locally integrable function e^{iax} for $a \in \mathbb{R}$. For any $\phi \in \mathcal{S}$, we have

$$\widehat{e^{iax}}(\phi) = e^{iax}\left(\hat{\phi}\right) = \int_{\mathbb{R}^d} e^{iax}\, \hat{\phi}(x)\, \mathrm{d}x = (2\pi)^{n/2}\, \phi(a) = (2\pi)^{n/2}\, \delta_a(\phi). \tag{44}$$

Therefore

$$\widehat{e^{iax}} = (2\pi)^{n/2}\, \delta_a. \tag{45}$$

The following properties are useful.

Proposition 24. *Let $D_j = -i\frac{\partial}{\partial x_j}$. Then*

- *a)* $\widehat{D_j T} = \xi_j\, \hat{T}$; *This implies* $\widehat{P(D) T} = P(\xi) \hat{T}$ *for any partial differential operator $P(D)$ whose coefficients are constants.*
- *b)* $\widehat{x_j T} = -D_j \hat{T}$;
- *c)* $\widehat{T * S} = (2\pi)^{n/2}\, \hat{T} \hat{S}$, *here we require $S \in \mathcal{E}'$*[7].

Example 25. Let $P(D)$ be a differential operator with constant coefficients (here we use D to denote $-i\partial$), then for any u,

$$\widehat{P(D) u} = P(\xi)\, \hat{u}. \tag{46}$$

6. Remember that the Fourier transform of an integrable function f is defined as

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\, \mathrm{d}x. \tag{39}$$

with inverse transform:

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} \hat{f}(\xi)\, \mathrm{d}\xi. \tag{40}$$

7. *This actually makes $\hat{S} \in C^\infty$. Recall that the multiplication between a C^∞ function and a distribution is well-defined.*

2. Existence for linear, constant-coefficient PDEs.

Let $P(D)$ be a differential operator of order m with constant coefficients (Here we define $D_j = -i\,\partial_{x_j}$). We try to solve the equation

$$P(D)u = f \tag{47}$$

for $f \in \mathcal{S}'$. We know that it suffices to find a fundamental solution G which solves

$$P(D)G = \delta. \tag{48}$$

Now we take the Fourier transform of both sides, and obtain

$$P(\xi)\,\hat{G} = \widehat{P(D)G} = \hat{\delta} = (2\,\pi)^{-n/2}. \tag{49}$$

Dividing both sides by $P(\xi)$ we have

$$\hat{G} = \frac{(2\,\pi)^{-n/2}}{P(\xi)}. \tag{50}$$

It suffices to show that $\hat{G} \in \mathcal{S}'$, or equivalently, $(P(\xi))^{-1} \in \mathcal{S}'$.

The general proof is complicated. Here we will present the idea of an elementary proof for a special case, that is when f has compact support, following Michael Taylor PDE §3.10. The main steps are the following.

1. By choosing a period box large enough, we can extend f periodically. After re-scaling we reduce the problem to the torus $\mathbb{T}^d$.
2. In this case the Fourier transform becomse Fourier series, and $\mathcal{D}' = \mathcal{S}'$.
3. $G \in \mathcal{D}'(\mathbb{T})$ if and only if $G(k)$ grows polynomially.
4. Observe that the solvability of $P(D)u = f$ is the same as $P(D+\alpha)v = g$ where $v = e^{-i\alpha\cdot x}\,u$ and $g = e^{-i\alpha\cdot x}\,f$. Thus we only need one α such that $P(k+\alpha)^{-1}\,\langle k\rangle^{-m} < \infty$ for some m. It suffices to show $P(k+\alpha)^{-\delta}\,\langle k\rangle^{-m} < \infty$ for some $\delta > 0$, $m > 0$.
5. For a polynomial $P(\xi)$ not identically zero, its zeroes are curves. One can show that there is $\delta_1 > 0$ such that
$$\int_{\mathbb{R}^d} |P(\xi)|^{-\delta} < \infty. \tag{51}$$
for all $0 < \delta < \delta_1$.
6. Let $Q(\xi) = |\xi|^{2m}\,P\left(\frac{\xi}{|\xi|^2}\right)$. Then Q is a polynomial and we obtain δ_2. Take $\delta = \min(\delta_1, \delta_2)$.
7. We have
$$\begin{aligned}\int_{|\xi|\geqslant 1} |P(\xi)|^{-\delta}\,|\xi|^{-2d}\,\mathrm{d}\xi &= \int_{|\eta|\leqslant 1} \left|P\left(\frac{\eta}{|\eta|^2}\right)\right|^{-\delta}\mathrm{d}\eta \\ &= \int_{|\eta|\leqslant 1} |\eta|^{2m\delta}\,|Q(\eta)|^{-\delta}\,\mathrm{d}\eta \\ &\leqslant \int_{|\eta|\leqslant 1} |Q(\eta)|^{-\delta}\,\mathrm{d}\eta < \infty.\end{aligned} \tag{52}$$
8. Therefore
$$\int_{0\leqslant\alpha_i\leqslant 1} \sum_{k\in\mathbb{Z}^d} |P(k+\alpha)|^{-\delta}\,\langle k\rangle^{-2d} \leqslant \int_{\mathbb{R}^d} |P(\xi)|^{-\delta}\,|\xi|^{-2d}\,\mathrm{d}\xi < \infty. \tag{53}$$

   Consequently we can find α such that the desired bound holds.

Remark 26. From the proof we see that when $f \in C_0^\infty$, the solution u also belongs to C_0^∞.

3. Elliptic PDEs with constant coefficients.

Recall that $P(D)$ is elliptic if and only if

$$|P(\xi)| \geqslant C\,|\xi|^m \tag{54}$$

for large ξ. We consider

$$P(D)u = f \tag{55}$$

for such $P(D)$. It turns out that we can find $E \in \mathcal{S}'$ which is smooth everywhere except the origin, rapidly decreasing at ∞, such that

$$P(D)E = \delta + w \tag{56}$$

where $w \in \mathcal{S}(\mathbb{R}^d)$. This E is called a parametrix for $P(D)$.

Now we have

$$u = (\delta + w) * u - w * u = (P(D)E) * u - w * u = E * (P(D)u) - w * u = E * f - w * u. \tag{57}$$

Definition 27. (Singular support) *The singular support of a general distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is the smallest set K such that u is smooth on $\mathbb{R}^d \backslash K$ (read: u can be identified with a smooth function on $\mathbb{R}^d \backslash K$). This set is denoted*

$$\operatorname{sing\,supp} u. \tag{58}$$

Example 28. We have

$$\operatorname{sing\,supp} \delta = \{0\}. \tag{59}$$

One can show that $\operatorname{sing\,supp}(f * g) \subset \operatorname{sing\,supp}(f) + \operatorname{sing\,supp}(g)$. In particular, for the parametrix E, we have

$$\operatorname{sing\,supp}(E * f) \subset \operatorname{sing\,supp} f. \tag{60}$$

Since $w * u \in C^\infty$, we have established

Proposition 29. *For any $u \in \mathcal{D}'(\mathbb{R}^d)$ solving $P(D)u = f$, if $P(D)$ is elliptic, then*

$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp} f. \tag{61}$$

Further readings.

- Michael Taylor, **Partial Differential Equations**, Vol. 1, Chap. 3, Springer-Verlag.
- Joel Smoller, **Shock Waves and Reaction-Diffusion Equations**, Chap. 7, Springer-Verlag.

Exercises.

Exercise 1. Show that

$$\lim_{n \nearrow \infty} \sin(n\,x) = 0 \tag{62}$$

in $\mathcal{D}'(\mathbb{R})$, but that

$$\lim_{n \nearrow \infty} \sin^2(n\,x) \neq 0. \tag{63}$$

This is the paradigm example in understanding weak convergence and homogenization theory. Hint: the first part follows directly from the so-called Riemann-Lebesgue Lemma.

Exercise 2. Find the distributional derivatives of the following functions.

1. $f(x) = \ln|x|$, $x \in \mathbb{R}$.
2. $f(x) = \begin{cases} f_1(x) & x > 0 \\ f_2(x) & x < 0 \end{cases}$, $x \in \mathbb{R},\ f_1, f_2 \in C^\infty$.

Exercise 3. Show that the general solution of $x\,y' = 0$ is $c_1 + c_2\,H(x)$ where H is the Heaviside function. Note that $x\,y'$ is the product of a C^∞ function and a distribution. Hint: First show that for $\phi \in C_0^\infty$ which vanishes at the origin, $\phi/x \in C_0^\infty$ too, then show that $y' = c\,\delta$ for some constant c.

Exercise 4. Let $u(x, t) = f(x + t)$ where f is any locally integrable function on $\mathbb{R}$. Then $u \in \mathcal{D}'(\mathbb{R}^2)$. Show that u solves the wave equation $u_{t\,t} - u_{x\,x} = 0$ in the sense of distributions.

Exercise 5. Let $\phi \in C_0^\infty(\mathbb{R})$ satisfy $\int_{\mathbb{R}} \phi = 1$. Define

$$\phi_n = n\,\phi(n\,x). \tag{64}$$

Prove that $\phi_n \to \delta$ in $\mathcal{D}'$.

Exercise 6. Prove that if ϕ vanishes in a neighborhood of the support of $T \in \mathcal{D}'$, then $T(\phi) = 0$. Would it suffice if ϕ vanishes on the support of T? Hint: use "partition of unity".