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The Cauchy-Kowalevski Theorem

In this lecture we review the existence and uniqueness results for general nonlinear PDEs in $\mathbb{R}^d$, with prescribed data on a hypersurface. In this case, the Cauchy-Kowalevski Theorem guarantees welll-posedness when the the data (the coefficients, the values of the unknown functions and its derivatives on the surface, and the surface iteslt) is analytic. It turns out that natural generalizations of this result are not possible.

1. The Cauchy-Kowalevski Theorem.

We use the multi-index notation introduced by L. Schwartz to state the theorem. A multi-index is a vector

$$\alpha = (\alpha_1, \ldots, \alpha_d) \tag{1}$$

where each α_i is a non-negative integer. The notation $\alpha \geqslant \beta$ means $\alpha_i \geqslant \beta_i$ for every i. For any multi-index α, we denote

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d, \qquad \alpha! = \alpha_1! \cdots \alpha_d!. \tag{2}$$

For any vector $\boldsymbol{x} = (x_1, \ldots, x_d)$, we denote

$$\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}. \tag{3}$$

We further write

$$\partial^\alpha = D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}. \tag{4}$$

Thus for example, when $\alpha = (1, 2, 1)$, $D^\alpha u$(or $\partial^\alpha u$) is the short-hand for $\frac{\partial^4 u}{\partial x_1 \, \partial x_2^2 \, \partial x_3}$.

Theorem 1. (Cauchy-Kowalevski) *Consider the initial value problem*

$$\partial_t^m u = G\Big(t, x, \partial_t^j \partial_x^\alpha u;\ 0 \leqslant j \leqslant m-1,\ j + |\alpha| \leqslant m\Big), \quad \partial_t^j u(0, \cdot) = g_j(\cdot) \quad 0 \leqslant j \leqslant m-1. \tag{5}$$

Suppose that g_j is real analytic on a neighborhood of $x_0 \in \mathbb{R}^d$, and G is real analytic on a neighborhood of $\Big(0, x_0, \partial_t^j \partial_x^\alpha g_j(x_0);\ j \leqslant m-1, j + |\alpha| \leqslant m\Big)$. Then there is a real analytic solution defined on an $\mathbb{R} \times \mathbb{R}^d$ neighborhood of $(0, x_0)$. The solution is unique in the sense that if u and v are both real analytic solutions to the equation on a connected neighborhood of $(0, x_0)$, then $u \equiv v$.

Remark 2. For fully nonlinear PDEs, one can apply implicit function theorem first to reduce the equation to the above form.

Before the 20th century, the dominating method of solving PDEs is power series expansion, with the underlying assumption that any "realistic" solution must have a convergent power series expansion (that is analytic) in some neighborhood of the initial point/surface. However, later people realized that analyticity is not an appropriate requirement at all, and became interested in non-analytic solutions. The following questions are naturally asked.

1. Does the theorem still hold when analyticity of the coefficients is replaced by differentiability? In particular, does a PDE with C^∞ data in general has unique C^∞ solution?

2. According to the C-K theorem, there is only one analytic solution; But can there be non-analytic solutions for an analytic PDE? In other words, does uniqueness still hold if we look at bigger solution spaces?

The answer to the first question is no. For the second question, when the equation is linear, Holmgren's theorem guarantees that the answer is no; When the equation is nonlinear, the problem is still unsettled. We will discuss the two questions briefly after presenting the main ideas of the proof of the C-K theorem.

2. Idea of the Proof.

We present two examples, one linear PDE and one nonlinear ODE, to illustrate the basic ideas of the proof.

Example 3. Show that the following PDE has a solution around the origin.

$$u_t - i\, u_x = 0, \qquad u(0, \cdot) = g(\cdot) \tag{6}$$

Solution. We try to construct a Taylor series for the two-variable function $u(t, x)$,

$$\sum \frac{\partial_t^j \partial_x^k u(0,0)}{j!\, k!}\, t^j\, x^k, \tag{7}$$

and then show its convergence. Note that the uniqueness part is trivial.

- $u(0,0) = g(0)$;
- $\partial_x u(0,0) = g'(0)$;
- $\partial_t u(0,0) = i\, \partial_x u(0,0) = i\, g'(0)$;
- $\partial_x^2 u(0,0) = g''(0)$;
- $\partial_t\, \partial_x u(0,0) = i\, \partial_x^2 u(0,0) = i\, g''(0)$;
- $\partial_t^2 u(0,0) = i\, \partial_t\, \partial_x u(0,0) = g''(0)$;
- $\cdots$.

In general, we have

$$\partial_t^j \partial_x^k u(0,0) = -\, i\, \partial_t^{j-1} \partial_x^{k+1} u(0,0) = \cdots = i^j\, \partial_x^{i+j} u(0,0) = i^j\, g^{(j+k)}(0). \tag{8}$$

Thus we obtain the Taylor series

$$\sum_{j,k=0}^{\infty} \frac{i^j\, g^{(j+k)}(0)}{j!\, k!}\, t^j\, x^k. \tag{9}$$

Now since g is analytic, there is $R > 0$ such that the Taylor series

$$\sum_{l=0}^{\infty} \frac{g^{(l)}(0)}{l!}\, y^l \tag{10}$$

converges for all $|y| \leqslant R$. This implies the existence of a constant C such that

$$\left| g^{(l)}(0) \right| \leqslant C\, \frac{l!}{R^l}. \tag{11}$$

Setting $l = j + k$ we see that the series for u is dominated by

$$C \sum_{j,k=0}^{\infty} \frac{(j+k)!}{j!\, k!} \left| \frac{t}{R} \right|^j \left| \frac{x}{R} \right|^k \tag{12}$$

which can be re-ordered into

$$C \sum_{m=0}^{\infty} \sum_{j+k=m} \frac{m!}{j!\, (m-j)!} \left| \frac{t}{R} \right|^j \left| \frac{x}{R} \right|^{m-j} = C \sum_{m=0}^{\infty} \left(\frac{|t| + |x|}{R} \right)^m. \tag{13}$$

It is clear that convergence holds when $|t| + |x| < R$.

For nonlinear equations, it can be hard to obtain formulas for the coefficients and estimate their sizes (interested readers can try to do this for the next example). Cauchy discovered a elegant way to overcome this difficulty. We illustrate his "majorization method" via a nonlinear ODE.

Example 4. Show that the following ODE

$$\dot{u} - u^3 = 0, \qquad u(0) = 1. \tag{14}$$

has a solution in the neighborhood of the origin.

Solution. We try to construct the unknown function u by computing its derivatives at the initial time 0. then formally we have

$$u(t) = u(0) + u'(0)\, t + \frac{1}{2}\, u''(0)\, t^2 + \cdots \tag{15}$$

This formal relation becomes rigorous when the power series on the RHS converges in some neighborhood of 0.

The first two terms in the series are already known. It turns out that we can compute the remaining terms successively using the initial values and the equation:

$$\begin{aligned} u'(0) &= u(0)^3 = 1; &(16)\\ u''(0) &= 3\, u(0)^2\, u'(0) = 3; &(17)\\ u^{(3)}(0) &= 3\, u(0)^2\, u''(0) + 6\, u(0)\, u'(0)^2 = 15; &(18)\\ &\vdots \end{aligned}$$

Then it is clear that we only need to show the convergence of the series

$$1 + t + \frac{3}{2}\, t^2 + \frac{15}{3!}\, t^4 + \cdots \tag{19}$$

However it is not clear how to directly bound these coefficients. The "method of majorants" is a somewhat indirect way to obtain the bounds. It is actually better illustrated when we consider the general case

$$\dot{v} = f(v), \qquad v(0) = 0. \tag{20}$$

The idea is to find a "majorant" $F(\,\cdot\,)$ such that $F^{(j)}(0) \geqslant \left| f^{(j)}(0) \right|$ for all $j \geqslant 0$. Then it is clear that the infinite series solution for the auxiliary equation

$$\dot{V} = F(V), \qquad V(0) = 0 \tag{21}$$

dominates that of the orginal equation. If we can find $F(\,\cdot\,)$ such that a real analytic solution for $\dot{V} = F(V)$ can be found explicitly, then we are done.

Since f is analytic, there is $C > 0$ such that

$$\left| f^{(k)}(0) \right| \leqslant C\, k!\, r^{-k} \tag{22}$$

for some $r > 0$. It turns out that the function $F(V) \equiv \frac{C\, r}{r - V}$ gives exactly

$$F^{(k)}(0) = C\, k!\, r^{-k}. \tag{23}$$

Thus all we need to do is to show that the ODE

$$\dot{V} = \frac{C\, r}{r - V}, \qquad V(0) = 0 \tag{24}$$

has a real analytic solution in the neighborhood of the origin. This ODE can be solved explicitly:

$$V(t) = r - \sqrt{r^2 - 2\, C\, r\, t}. \tag{25}$$

(The other possibility $V(t) = r + \sqrt{r^2 - 2\, C\, r\, t}$ is discarded because $\dot{V} > 0$ according to the equation) This function is analytic in a neighborhood of 0.

Uniqueness is obvious since all the derivatives are uniquely determined.

Remark 5. The above general result for ODEs is called "Cauchy's Theorem". It is said that Cauchy is the first mathematician who systematically considered the problem of whether such formal series arising from solving ODEs and PDEs are convergent.

3. General Cauchy Problems and Characteristics.

The general Cauchy problem is the following. Consider a partial differential equation of order m:

$$F(x, \partial^\alpha u;\ |\alpha| \leqslant m) = 0, \tag{26}$$

and a smooth hypersurface, Σ in $\mathbb{R}^d$, given by

$$\phi(x_1, \ldots, x_d) = 0. \tag{27}$$

Here ϕ shall have m continuous derivatives and the surface should be regular in the sense that

$$D\phi = (\phi_{x_1}, \ldots, \phi_{x_d}) \neq \mathbf{0}. \tag{28}$$

Now the task is to solve the equation around any given point $\boldsymbol{x}_0 \in \Sigma$.

When ϕ is analytic, one can "flatten" Σ through an analytic change of variables $\boldsymbol{x} \to \boldsymbol{y}$ such that Σ in $\boldsymbol{y}$-space becomes $y_n = 0$. At the same time the equation becomes $G(\boldsymbol{y}, \partial^\alpha u;\ |\alpha| \leqslant m) = 0$. From the proof of the Cauchy-Kowalevski theorem we know that to be able to construct the solution, a sufficient condition is $\frac{\partial G}{\partial(\partial_{y_n}^m)} \neq 0$. It turns out that this condition is the same as

$$\lim_{\lambda \to \infty} \frac{e^{-i\lambda\phi} P(x, \partial) e^{i\lambda\phi}}{\lambda^m} \neq 0 \qquad \text{at } \boldsymbol{x}_0. \tag{29}$$

Here

$$P = \sum a_\alpha(x)\, \partial^\alpha, \qquad a_\alpha(x) = \frac{\partial F}{\partial(\partial^\alpha u)}\left(x, \partial^\beta u\right) \tag{30}$$

is the linearization of the original nonlinear PDE. Such a hypersurface Σ is called *noncharacteristic* at $\boldsymbol{x}_0$. It is called *characteristic* at $\boldsymbol{x}_0$ when the condition is not satisfied.

Example 6. For a linear operator $P = \sum_{|\alpha| \leqslant m} a_\alpha\, \partial^\alpha$, a surface $\phi = 0$ is noncharacteristic if and only if

$$\sum_{|\alpha| = m} a_\alpha\, (D\phi)^\alpha \neq 0. \tag{31}$$

Note that the lower order terms do not matter.

For example, consider the Laplacian

$$\triangle = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 \tag{32}$$

the condition for ϕ to be noncharacteristic is

$$(\partial_{x_1}\phi)^2 + \cdots + (\partial_{x_d}\phi)^2 \neq 0. \tag{33}$$

When the unknown function is a vector, each coefficient of the operator P is a matrix: $P = \sum_{|\alpha| \leqslant m} A_\alpha\, \partial^\alpha$, the condition for ϕ to be noncharacteristic is

$$\det\left(\sum_{|\alpha| = m} A_\alpha\, (D\phi)^\alpha \right) \neq 0. \tag{34}$$

Definition 7. (Symbol of a differential operator) *For a linear differential operator* $P(x,\ \partial) = \sum_{|\alpha| \leqslant m} A_\alpha\, (-i\,\partial)^\alpha$*, we define its principal part as*

$$L_p = \sum_{|\alpha| = m} A_\alpha\, (-i\,\partial)^\alpha. \tag{35}$$

The "symbol" of this operator is the matrix form

$$\Lambda(\xi) \equiv \sum_{|\alpha| = m} A_\alpha \xi^\alpha. \tag{36}$$

where $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix}$.

We can further define the *characteristic form*

$$Q(\xi) = \det(\Lambda(\xi)). \tag{37}$$

Thus a surface $\phi = 0$ is noncharacteristic if and only if

$$Q(D\phi) \neq 0. \tag{38}$$

Definition 8. *An linear differential operator is called "elliptic" if its characteristic form does not vanish for any* $\xi \neq \mathbf{0}$.

In other words, for an elliptic operator, any hypersurface is noncharacteristic. This fact has great effect on the property of solutions. For example, it can be shown that if the solution u is piecewise C^∞, then the hypersurface across which u is not C^∞ must be characteristic. As a consequence, an elliptic PDE cannot have such solutions. On the other hand, as we will see in future lectures, such solutions appear in hyperbolic PDEs.

Example 9. The Laplacian operator is elliptic, while the heat operator and the wave operator are not.

4. Lewy's Example, Holmgren's Theorem, and Other Remarks.

4.1. Lewy's counter-example.

It is natural to ask whether a Cauchy-Kowalevski type theorem can hold for PDEs with non-analytic coefficients. It is relatively easy to construct examples of Cauchy problems where the analyticity of initial values is crucial. The importance of the coefficients is made clear by H. Lewy[1] when he constructed a PDE with C^∞ coefficients but does not allow any C^1 solution anywhere.

Example 10. Consider the PDE

$$u_t - i\, u_x = 0 \tag{39}$$

with initial data $g(x)$ prescribed on $t = 0$. By the Cauchy-Kowalevski theorem, we know that when g is analytic, there is an analytic solution in a neighborhood of any $(0, x_0)$.

Now consider the general case. We show that when g is not analytic, there can be no C^1 solution. We prove by contradiction. Let $u(t, x) \in C^1$ solve the equation. Identify the (t, x) plane with $\mathbb{C}$ by denoting $z = x + i\,t$. Write $u = u(z) = \Re u + i\,\Im u$. Then from the equation we have

$$(\Re u)_x = (\Im u)_t, \qquad (\Im u)_x = -(\Re u)_t \tag{40}$$

which implies u is in fact analytic for $t > 0$ in a small neighborhood of $(0,\ x_0)$. Consequently g must be real analytic.

Example 11. (H. Lewy 1957) For a complex-valued function $u(x, y, z)$, consider the linear PDE

$$L u \equiv -u_x - i\, u_y + 2\, i\, (x + i\, y)\, u_z = f(x, y, z). \tag{41}$$

Then there are infinitely many $f \in C^\infty$ that the equation has no C^1 solution in any open subset $\Omega \subset \mathbb{R}^3$.

1. H. Lewy, *An example of a smooth linear partial differential equation without solution*. Ann. of Math. (2) 66 (1957), 155–158.

The idea is as follows. First one considers a special right hand side $\psi'(z)$ for a real function ψ, and shows that if u is C^1 in any neighborhood of the origin, then ψ must be real analytic around 0; Then one picks $\psi \in C^\infty$ but is nowhere real analytic[2], chooses all the rational points (x_i, y_i, z_i) and consider

$$f(x,y,z) = \sum c_i \psi'(z - 2 y_i x + 2 x_i y) \tag{43}$$

and show that existence of $\{c_i\}$ such that no C^1 solution is allowed anywhere. It turns out there are infinitely many sequences $\{c_i\}$ which satisfy this criterion.

Remark 12. An example of a PDE with real coefficients that does not allow any C^1 solution is given in a 6-line paper: F. Treves, *The equation* $\left(\partial^2/\partial x^2 + \partial^2/\partial y^2 + (x^2+y^2)(\partial/\partial t)\right)^2 u + \partial^2 u/\partial t^2 = f$, *with real coefficients, is "without solutions"*, Bull. Amer. Math. Soc., 68, 332–332, 1962.

4.2. Holmgren's uniqueness theorem.

By the Cauchy-Kowalevski theorem, we know that if the coefficients, initial surface, and initial values of a PDE are analytic, then there is a unique analytic solution around any initial point. A natural question to ask is whether there can be any other solution which is not analytic. Holmgren's theorem asserts that this is the case when the PDE is linear.

We will mention here an improved version, due to Fritz John:

Theorem 13. (Fritz John's Global Holmgren Theorem) *If $u \in C^m(\Omega)$, $Pu = 0$ in Ω and $\partial^\alpha u = 0$ on a noncharacteristic surface Σ for $|\alpha| \leqslant m-1$, then u vanishes on any set swept out by deforming Σ through noncharacteristic surfaces whose ends stay in Σ.*

Example 14. Let u solve an elliptic PDE. If u vanishes along a hypersurface Σ, then u vanishes everywhere.

Remark 15. When the equation is nonlinear, the order of the equation turns out to play an important role.

For first order PDEs, similar uniqueness theorem was proved in G. Metivier *Uniqueness and approximation of solutions of first order non linear equations*, Invent. math. 82, 263 – 282 (1985).

For higher order PDEs (or equivalently, systems), G. Metivier constructed an example of a nonlinear analytic system having two different C^∞ solutions. See G. Metivier, *Counterexamples to Hölmgren's uniqueness for analytic nonlinear Cauchy problems*, Invent. math. 112 (1993), no. 1, 217–222. Another counterexample is constructed by L. Hörmander in *A counterexample of Gevrey class to the uniqueness of the Cauchy problem*, Math. Res. Lett. 7 (2000), no. 5-6, 615–624.

Further reading.

More detailed presentation of the materials in this lecture can be find in many standard PDE textbooks. For example

- Fritz John, **Partial Differential Equations**, Springer-Verlag;
- Jeffrey Rauch, **Partial Differential Equations**, Springer-Verlag;
- Martin Schechter, **Modern Methods in Partial Differential Equations: An Introduction**.
- Michael Taylor, **Partial Differential Equations**, Vol. 1, Springer-Verlag.

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2. For example,

$$\psi(z) = \sum_1^\infty \frac{\cos(n!\,x)}{(n!)^n}. \tag{42}$$

To see that it's not analytic anywhere, first note that there are no C and r such that $\left|\psi^{(n)}(0)\right| \leqslant n!\,\frac{C}{r^n}$, which means ψ is not analytic at 0; Next note that $\psi(z) - \psi(z-q)$ is analytic for any $q \in \mathbb{Q}$, this implies ψ cannot be analaytic at any rational number q.

In fact, one can show that such "pathological" functions form a second category set in C^∞.

Exercises.

Exercise 1. Consider the wave operator in $\mathbb{R} \times \mathbb{R}$:

$$\partial_t^2 - \partial_x^2. \tag{44}$$

Determine its characteristic surfaces.

Exercise 2. Let u be a C^2 solution of the 1D wave equation

$$u_{tt} - u_{xx} = 0, \qquad u(0, x) = g(x), \;\; u_t(0, x) = h(x). \tag{45}$$

Assume that $g(x) = h(x) = 0$ for $x \in [-1, 1]$. Use Fritz John's theorem to determine the region where u must vanish.

Exercise 3. Consider the 1D heat equation

$$u_t - u_{xx} = 0, \tag{46}$$

Show that

$$u(t, x) = (4\pi t)^{-1/2} e^{-x^2/4t} \tag{47}$$

is a smooth solution in $t > 0$. Extend u by 0 to $t < 0$. Show that the resulting function is $C^\infty(\mathbb{R}^2 \backslash \{\mathbf{0}\})$ and satisfies the heat equation on $\mathbb{R}^2 \backslash \{\mathbf{0}\}$. Show that u does not vanish in any neighborhood of any point on the x-axis. Is this contradictory to the Holmgren Theorem?