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Introduction

1. Examples of PDEs.

1.1. PDEs from physics.

"PDE arose in the context of the development of models in the physics of continuous media, e.g. vibrating strings, elasticity, the Newtonian gravitational field of extended matter, electrostatics, fluid flows, and later by the theories of heat conduction, electricity and magnetism."[1]

Example 1. Laplace's equation.

$$\triangle u = 0. \tag{1}$$

where

$$\triangle \equiv \partial_{x_1 x_1} + \cdots + \partial_{x_d x_d}. \tag{2}$$

is the Laplacian. When the right hand side is not 0, that is

$$(-)\triangle u = f \tag{3}$$

It's called Poisson's equation.

- First studied by Laplace in his work on gravitational potential fields around 1780.

Example 2. Heat equation.

$$u_t - D \triangle u = 0. \tag{4}$$

Here D is the thermal diffusivity of the material and u is the (absolute) temperature. It models the convection of heat through some material.

- Introduced by Fourier in his memoir *Théorie analytique de la chaleur* (1810 − 1822).
- By writing the equation as

  $$u_t - \nabla \cdot (D \nabla u) = 0, \tag{5}$$

  it becomes a conservation law.

Example 3. Wave equation.

$$\Box u \equiv u_{tt} - \triangle u = 0. \tag{6}$$

Where the operator $\partial_{tt} - \triangle$ is often denoted $\Box$ and called the "D'Alembertian".

- $d = 1$: Introduced by d'Alembert in 1752 modeling a vibrating string.
- $d = 2, 3$: Euler (1759), Bernoulli (1762), small amplitude sound waves.

Remark 4. The above three equations are among the first PDEs studied and understood by mathematicians[2]. Some what mysteriously, the three operators $\triangle$, $\partial_t - \triangle$, and $\Box$ turned out to be ubiquitous. As a consequence, a solid understanding of them (and their generalizations) is fundamental in the study of PDEs.

Other examples of equations arising from physics are

Example 5. Linear transport equation.

$$u_t + b \cdot \nabla u = u_t + \sum_{i=1}^{d} b^i u_{x_i} = 0. \tag{7}$$

Here u can be the density of certain substance, and $b = b(\boldsymbol{x}, t)$ is the velocity of the "stream" which carries the substance around.

1. Brezis-Browder, *Partial differential equations in the 20th century*.

2. Many other equations were introduced around the same time, for example the Euler/Navier-Stokes equations modeling incompressible fluids and the Monge-Ampère equation arising from optimal planning of transportation. Most of these other equations turned out to be much harder than the three "basic" equations.

Example 6. Schrödinger's equation.

$$i\,\hbar\,u_t = -\frac{\hbar^2}{2\,m}\,\triangle u + V(\boldsymbol{x},\,u). \tag{8}$$

It describes the evolution of the probability density of a particle subject to a potential V.

Remark 7. The "transport" operator $\partial_t + b\cdot\nabla$, and the Schrödinger operator $-i\,\partial_t - \triangle$ are other examples of fundamental operators in PDE theory.

Example 8. Incompressible Navier-Stokes equations

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\,\triangle\boldsymbol{u}, \qquad \operatorname{div}\boldsymbol{u} = 0. \tag{9}$$

When the viscosity $\nu = 0$, we have the incompressible Euler equations.

- Note the appearance of the heat operator and the transport operator. In fact, most progress in the study of the Navier-Stokes equation comes from understanding of these two operators as well as the interaction between them.
- The term $-\nabla p$ appears as a Lagrangian multiplier of the constraint $\operatorname{div}\boldsymbol{u} = 0$.
- The Euler equations were proposed by L. Euler in 1755, the Navier-Stokes equations were proposed by Navier (1822–27), Poisson (1831), and Stokes (1845) as a more realistic model of incompressible fluids.
- The Navier-Stokes/Euler equations are still poorly understood today.[3]

Example 9. Boltzmann equation.

$$f_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f = Q(f,f) \qquad \boldsymbol{x}\in\mathbb{R}^d,\ \boldsymbol{v}\in\mathbb{R}^d,\ t\geqslant 0. \tag{10}$$

The unknown function f corresponds at each time t to the density of particles at the point $\boldsymbol{x}$ with velocity $\boldsymbol{v}$. $Q(f,f)$ represents interaction between particles. Q is a quadratic operator, the underlying assumption to this is that interaction between more than three particles can be neglected.

One can show that under natural assumptions, the Navier-Stokes equations can be obtained from the Boltzmann equation through averaging the velocity $\boldsymbol{v}$ out[4].

Example 10. 1D Gas Dynamics

$$v_t - u_x = 0 \tag{11}$$
$$u_t + p_x = 0 \tag{12}$$
$$S_t = 0 \tag{13}$$

where $v = \rho^{-1}$ with ρ the density, p is the pressure, u the velocity, and S the entropy. p is related to the other unknowns by the equation of state:

$$p = p(v,S). \tag{14}$$

This is an example of the general system of 1D conservation laws

$$\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0 \tag{15}$$

if we take $\boldsymbol{u} = \begin{pmatrix} v \\ u \\ S \end{pmatrix}$ and $\boldsymbol{F}(v,u,S) = \begin{pmatrix} -u \\ p(v,S) \\ 0 \end{pmatrix}$.

3. From Clay institute website (http://www.claymath.org/millennium/Navier-Stokes_Equations/): "Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations."

4. F. Golse, L. Saint-Raymond, *The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels*, Invent. Math. 155 (2004), 81 – 161.

Remark 11. Conservation laws is a major research area during the cold war, due to its applications in aerodynamics. On the other hand, methods and understanding developed through the study of conservation laws have been successfully applied to other PDEs and even other fields. In particular, numerical schemes designed for computing the solutions of conservation laws proved very useful in image processing.

Example 12. Korteweg-de Vries (KdV) equation

$$u_t - 6\,u\,u_x + u_{xxx} = 0. \tag{16}$$

- Introduced in 1896 as a model for solitary water waves.
- Can be derived from shallow water equation.
- Turned out to be related to algebraic geometry.
- Here we meet a new operator $\partial_t + \partial_{xxx}$. It is the fundamental operator modeling dispersive phenomena (in comparison, the heat operator $\partial_t + \partial_{xx}$ represents the diffusive phenomena). The two operators have totally different properties. [5]

Example 13. Maxwell's equations.

$$\begin{aligned} \boldsymbol{E}_t &= \operatorname{curl} \boldsymbol{B} \qquad (17)\\ \boldsymbol{B}_t &= -\operatorname{curl} \boldsymbol{E} \qquad (18)\\ \operatorname{div} \boldsymbol{E} &= 4\,\pi\,\rho \qquad (19)\\ \operatorname{div} \boldsymbol{B} &= -4\,\pi\,j \qquad (20)\end{aligned}$$

where $\boldsymbol{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$ is the electric field and $\boldsymbol{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$ is the magnetic induction field.

- Combining the Maxwell's equations and the Navier-Stokes equations one obtains the so-called Magneto-hydrodynamics equations (MHD), which plays an important role in understanding the local universe (for example the solar winds) around us.
- If we take the time derivative of the first two equations, we obtain

  $$\begin{aligned} \boldsymbol{E}_{tt} &= \operatorname{curl}(\operatorname{curl} \boldsymbol{E}) = \triangle \boldsymbol{E} + 4\,\pi\,\nabla \rho \qquad (21)\\ \boldsymbol{B}_{tt} &= \operatorname{curl}(\operatorname{curl} \boldsymbol{B}) = \triangle \boldsymbol{B} - 4\,\pi\,\nabla j \qquad (22)\end{aligned}$$

  and thus revealed the "hidden" wave equations in the system.

Example 14. The Einstein field equation of General Relativity for the curvature of the metric (g_{ij}) of space-time:

$$R_{ij} - \frac{1}{2}\,g_{ij}\,R = \kappa\,T_{ij} \qquad i,j = 0,1,2,3. \tag{23}$$

κ is a constant, T_{ij} is the energy-momentum tensor,

$$R_{ij} \equiv \sum_{k=0}^{3} \left(\frac{\partial}{\partial x^k} \Gamma_{ij}^k - \frac{\partial}{\partial x^j} \Gamma_{ik}^k + \sum_{l=0}^{3} \left(\Gamma_{lk}^k\,\Gamma_{ij}^l - \Gamma_{lj}^k\,\Gamma_{ik}^l \right) \right) \tag{24}$$

is the Ricci curvature. Here

$$\Gamma_{ij}^k \equiv \frac{1}{2} \sum_{l=0}^{3} g^{kl} \left(\frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right) \tag{25}$$

$$\left(g^{ij}\right) \equiv (g_{ij})^{-1} \tag{26}$$

and

$$R \equiv \sum_{i,j=0}^{3} g^{ij}\,R_{ij} \tag{27}$$

is the scalar curvature.

5. Anyone familiar with the theory of weak solutions for the viscous Burgers equation $u_t + u\,u_x - u_{xx} = 0$ can take a look at the three papers in the early 80s by P. D. Lax and C. D. Levermore to get some idea of how different the two operators are.

1.2. PDEs from other sciences.

Example 15. Scalar reaction-diffusion equation

$$u_t - \triangle u = f(u) \tag{28}$$

or reaction-diffusion systems

$$\boldsymbol{u}_t - \triangle \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}). \tag{29}$$

One can also put in a transport (or conservation law) term:

$$\boldsymbol{u}_t + \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \triangle \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}), \tag{30}$$

$$\boldsymbol{u}_t + \nabla \cdot (\boldsymbol{b} \otimes \boldsymbol{u}) - \triangle \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}). \tag{31}$$

Such equations appear in Chemistry and Mathematical Biology. The velocity $\boldsymbol{b}$ can be given or unknown. In the latter case $\boldsymbol{b}$ is often related to $\boldsymbol{u}$ via some nonlocal operator, for example

$$\boldsymbol{b}(\boldsymbol{x}, t) = \int \nabla K(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y}, t)\, \mathrm{d}\boldsymbol{y} \tag{32}$$

where $K(z) = e^{-|\boldsymbol{x}|}$[6] which has been extensively studied recently.

Example 16. Porous medium equation

$$u_t - \nabla \cdot (u^\gamma \nabla u) = 0. \tag{33}$$

where u is the density of the substance being transported in a porous medium. The above equation is in fact a conservation law $u_t - \nabla \cdot (\boldsymbol{b}\, u) = 0$ where the velocity $\boldsymbol{b}$ depends on u locally due to equation of state and the so-called Darcy Law governing motion of fluids in porous media.[7]

This equation can also be studied from the heat equation point of view (remember that the heat equation $u_t - D \triangle u = 0$ can be written as $u_t - \nabla \cdot (D \nabla u) = 0$). It indeed share many good properties with the heat equation (e.g. maximum principle).

Example 17. Black-Scholes equation:

$$u_t + \frac{1}{2} \sigma^2 x^2 u_{x x} + r\, x\, u_x - r\, u = 0. \tag{34}$$

Here $u = u(x, t)$, $x \geqslant 0, t \geqslant 0$ is the price of a European option.[8]

A second's inspection reveals that this equation is a combination of fundamental opeators.

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6. A. L. Bertozzi and T. Laurent, *Finite-time blow-up of solutions of an aggregation equation in* $\mathbb{R}^n$, Comm. Math. Phys., 274, p. 717-735, 2007.

7. See e.g. D. G. Aronson, *The porous medium equation*, Lecture Notes in Mathematics 1224, 1–45, 1986.

8. There is a expository article explaining this equation in Terence Tao's blog: http://terrytao.wordpress.com/2008/07/01/the-black-scholes-equation/.

1.3. PDEs from engineering.

Example 18. Monge-Ampère equation

$$\det\left(D^2 u\right) = f. \tag{35}$$

- First studied by Gaspard Monge in 1784 and later by André-Marie Ampère in 1820;
- The solution gives the optimal way of transporting material from one site to the other;
- It also arise naturally in several problems in Riemannian geometry, conformal geometry, and CR geometry. In particular, a surface $u = u(x,\, y)$ with prescribed Gauss curvature $K(x, y)$ satisfies

$$\det\left(D^2 u\right) = f \equiv K(x, y)\left(1 + u_x^2 + u_y^2\right)^2. \tag{36}$$

Remark 19. The operator $\det\left(D^2 \cdot\ \right)$ seems unrelated to any of the previous operators. However in fact there is the following link between the Monge-Ampere equation and the Poisson equation, which can be written as

$$\mathrm{tr}\left(D^2 u\right) = f. \tag{37}$$

Thus the Monge-Ampère equation is $\Pi \lambda_i = f$ while the Poisson equation is $\sum \lambda_i = f$, where λ_i's are the eigenvalues of the matrix $D^2 u$. From this view-point, it makes sense to study equations obtained by taking other invariants of the matrix $D^2 u$. Interestingly, all these equation share many common properties.

1.4. PDEs from other branches of mathematics.

Example 20. The Cauchy-Riemann equation

$$f_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\,\partial_y\right) f = 0. \tag{38}$$

If we write $f = u + i\,v$, then the equation becomes a system

$$u_x = v_y, \quad u_y = -v_x. \tag{39}$$

Now assuming $u, v \in C^2$ and taking derivatives, we arrive at

$$\triangle u = \triangle v = 0. \tag{40}$$

Example 21. Minimal surface equation.

$$\nabla \cdot \left(\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}}\right) = 0. \tag{41}$$

It is obtained as the Euler-Lagrange equation of minimizing the area of a surface given by $u = u(\boldsymbol{x})$:

$$S(u) = \int \left(1 + |\nabla u|^2\right)^{1/2} \mathrm{d}\boldsymbol{x}. \tag{42}$$

Example 22. Ricci flow.

$$g_t = -2\,\mathrm{Ric}, \qquad g(0) = g_0 \tag{43}$$

where g is a Riemannian metric and Ric is the so-called Ricci curvature.

- g evolves toward a uniform metric after re-scaling;
- This equation has some similar properties with the heat equation.
- The solution of the Poincaré Conjecture is based on understanding of this equation.

2. General formulation and terminologies.

In general, a PDE is an equation of the form

$$F\big(D^m u(x), D^{m-1}u(x), \ldots, Du(x), u(x), x\big) = 0 \qquad x \in \Omega \subseteq \mathbb{R}^d \tag{44}$$

where u is the unknown (can be scalar or vector) and $D^k u$ is a shorthand for the collection of all kth derivatives of u. The highest order of the derivatives involved is called the "order" of the PDE.

- A PDE is called *linear* if F is linear in u and its derivatives. In this case it has the form

$$\sum_{|\alpha| \leqslant m} a_\alpha(x)\, D^\alpha u = f(x). \tag{45}$$

  Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d}$.

- A PDE is called *quasilinear* if F is linear in the highest order derivatives of u, or equivalently, the equation has the form

$$\sum_{|\alpha| = m} a_\alpha\big(D^{k-1}u, \ldots, Du, u, x\big)\, D^\alpha u + a_0\big(D^{k-1}u, \ldots, Du, u, x\big) = 0. \tag{46}$$

  When all a_αs only depend on x, the PDE is called *semilinear*. The general form for semilinear PDEs is

$$\sum_{|\alpha| = m} a_\alpha(x)\, D^\alpha u + a_0\big(D^{k-1}u, \ldots, Du, u, x\big) = 0. \tag{47}$$

- A PDE is called *fully nonlinear* if it depends nonlinearly upon the highest order derivatives.

3. Well-posedness.

The well-posedness of a differential equation (ODE, PDE) has three aspects.

a) Existence: The problem has a solution.

b) Uniqueness: There is only one solution.

c) Stability: The solution depends continuously on the data given in the problem.

Remark 23. c) is especially important when the equation has its origin in other sciences.

Remark 24. In the following lectures we will see that one can define the meaning of "a function u solves the PDE" in more than one ways. There are two criteria to judge which definition we should take:

1. How relevant is the solution to the original physical (chemical, biological, etc.) process/phenomenon which leads to the PDE;

2. Does this definition of solutions makes the PDE well-posed.

The ideal situation (and people's expectation) is that one (the right) definition will simultaneously satisfy 1. and 2.. An example is the theory of conservation laws.

For difficult PDEs which still confuse researchers, different mathematicians may have different ideas as to which of 1 and 2 should guide the study.

Remark 25. The study of well-posedness of nonlinear equations naturally leads to the study of regularity of linear PDEs with non-constant coefficients.

Further reading.

The following are survey articles of PDE theory as a whole.

- Haïm Brezis, Felix Browder, *Partial differential equations in the 20th century*, Advances in Mathematics 135 (1998) 76-144.
- Luis Nirenberg, *Partial differential equations in the first half of the century*, in **Development of Mathematics: 1900 − 1950**, Jean-Paul Pier ed., Birkhäuser.
- Sergiu Klainerman, *PDE as a unified subject*, GAFA, Geom. funct. anal., Special Volume, 2000.

Exercises.

Exercise 1. List two PDEs, either from the above examples or from your own reading, that interest you most. Give a concise introduction (where does the equation come from, main ingredients of the derivation of the equation, why is the equation important, etc.) for each. **(5 pts for each PDE)**

Exercise 2. (Well-posedness for ODE) We develop a complete theory of well-posedness for the initial value problem of ODE. Consider an ODE of the form

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0. \tag{48}$$

where f is defined on $D \subseteq \mathbb{R} \times \mathbb{R}^d$ and $(t_0, u_0) \in D$. We say u is a classical solution if $u \in C^1$.

a) **(2 pts)** Existence I: Prove the following theorem.

Theorem. *Assume that f is continuous in t and uniformly Lipschitz in u, then there exists an interval $(t^-, t^+) \ni t_0$, such that at least one classical solution $u \in C^1(t^-, t^+)$ exists.*

Remark. The proof still works when $\mathbb{R}^d$ is replaced by any Banach space. As a consequence, it can be applied to many PDEs.

b) **(Optional)** Existence II: Prove the following theorem.

Theorem. *The "uniform Lipschitz" condition on f in the above theorem can be replaced by $f \in C(D)$.*

Hint: On any compact subset of D, approximate f uniformly by Lipschitz functions f_n, let u_n be a solution of the corresponding ODE, then use Ascoli-Arzela Theorem (a uniformly bounded, equicontinuous sequence has a subsequence which converges uniformly).

c) Uniqueness:

i. **(2 pts)** Show that the solution obtained in a) is in fact the only solution for the initial value problem.

ii. **(2 pts)** Construct an example to show that under the condition of the theorem in b), uniqueness may fail.

iii. **(Optional)** Show that uniqueness still holds when the "uniform Lipschitz" condition on f in a) is replaced by the following weaker "Osgood" condition:

$$|(f(t, u) - f(t, v)) \cdot (u - v)| \leqslant g(|u - v|) \tag{49}$$

where the modulus g satisfies

$$\int_0^\delta \frac{1}{g(r)} \,\mathrm{d}r = \infty \tag{50}$$

for any $\delta > 0$.

d) **(2 pts)** Continuous dependence on initial value:

Prove that the unique solution obtained in a) depends continuously on (t_0, u_0). Note that continuous dependence on data automatically fails when the solution is not unique.

e) **(2 pts)** Different definitions of solution, regularity:

One can integrate and obtain the following "weak" formulation

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s)) \,\mathrm{d}s. \tag{51}$$

We say $u \in C(I)$ is a "weak solution" of the ODE if it satisfies this integral formulation. Prove that, $u \in C^m$ if $f \in C^{m-1}$ (as a function of (t, u)) for $m \geqslant 1$.