Burnside's lemma

- Burnside's Lemma:
  - X: A collection of objects;
  - G: A group of operations on X;
  - If two objects $x, y \in X$ are viewed as the same when $y = g\,x$ for some $g \in G$, then the number of "truly different" objects is
$$\frac{1}{|G|}\sum_{g\in G} |X_g| \tag{1}$$
where $X_g = \{x \in X|\; g\,x = x\}$.
- Generalizes Polya's theory; Applies to more problems.

Example 1. How many ways are there to put 10 identical balls into 3 identical boxes, such that none of the boxes has more than 5 balls?

Solution. We mark the boxes $1, 2, 3$ and set

$$X := \{\text{Different ways putting 10 identical balls into 3 different boxes so that no box has more than 5}\}. \tag{2}$$

The symmetry group is

$$S_3 = \{(1)(2)(3), (12)(3), (13)(2), (23)(1), (123), (132)\}. \tag{3}$$

We calculate:

- $|X_i| = |X| =$ the number of solutions to $x_1 + x_2 + x_3 = 10, 0 \leqslant x_i \leqslant 5$. We solve it by the method of generating functions. As the balls are identical, the generating function is ordinary,
$$\begin{aligned} A(x) &= (1 + x + \cdots + x^5)^3 \\ &= \frac{(1-x^6)^3}{(1-x)^3} \\ &= (1 - 3\,x^6 + 3\,x^{12} - x^{18}) \sum_{n=0}^{\infty} \frac{(n+2)\,(n+1)}{2}\,x^n. \end{aligned} \tag{4}$$
The coefficient for x^{10} is then
$$\frac{(10+2)\,(10+1)}{2} - 3\,\frac{(4+2)\,(4+1)}{2} = 66 - 45 = 21. \tag{5}$$
- $|X_{(12)(3)}| =$ the number of solutions to $x_1 + x_2 + x_3 = 10, 0 \leqslant x_i \leqslant 5$ that is unchanged after switching $x_1 \longleftrightarrow x_2$. Clearly this means $x_1 = x_2$. So $|X_{(12)(3)}| =$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $0 \leqslant x_i \leqslant 5, x_1 = x_2$. This is simple enough to just count, but let's solve it by generating functions.

  The problem is equivalent to
$$2\,y_1 + y_2 = 10, \qquad 0 \leqslant x_i \leqslant 5 \tag{6}$$
which in turn is equivalent to
$$z_1 + z_2 = 10, \qquad 0 \leqslant z_1 \leqslant 10, z_1 \text{ even}, 0 \leqslant z_2 \leqslant 5. \tag{7}$$
The generating function is then
$$\begin{aligned} A(x) &= (1 + x^2 + x^4 + \cdots + x^{10})\,(1 + x + \cdots + x^5) \\ &= \frac{1-x^{12}}{1-x^2}\,\frac{1-x^6}{1-x} \\ &= (1-x^{12})\,(1-x^6)\,\frac{1}{(1+x)\,(1-x)^2}. \end{aligned} \tag{8}$$
We apply the method of partial fraction:
$$\frac{1}{(1+x)\,(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} \tag{9}$$
which leads to
$$1 = A\,(1-x^2) + B\,(1+x) + C\,(1-x)^2. \tag{10}$$

Setting $x=1$ we have $B=1/2$. Setting $x=-1$ we have $C=1/4$. Finally comparing the constant terms we have $1=A+B+C$ so $A=1/4$. Therefore

$$A(x)=(1-x^6-x^{12}+x^{18})\left[\frac{1}{4}\sum_{n=0}^{\infty} x^n+\frac{1}{2}\sum_{n=0}^{\infty} (n+1)\,x^n+\frac{1}{4}\sum_{n=0}^{\infty} (-1)^n\,x^n\right]. \tag{11}$$

The coefficient of x^{10} is then

$$\left[\frac{1}{4}+\frac{11}{2}+\frac{1}{4}\right]-\left[\frac{1}{4}+\frac{5}{2}+\frac{1}{4}\right]=3. \tag{12}$$

- Clearly $|X_{(23)(1)}|=|X_{(13)(2)}|=|X_{(12)(3)}|=3$.
- $|X_{(123)}|=$ the number of solutions to $x_1+x_2+x_3=10$, $0\leqslant x_i\leqslant 5$ that is unchanged after switching $x_1\rightarrow x_2, x_2\rightarrow x_3, x_3\rightarrow x_1$. Clearly this requires $x_1=x_2=x_3$. So $|X_{(123)}|=0$.
- $|X_{(132)}|=0$ for the same reason.

Therefore the answer is $\frac{21+3+3+3+0+0}{6}=5$ different ways.

Solving recurrence relations

Note. Only ordinary generating functions will be involved in the final exam regarding recurrence relations.

Example 2. Solve $h_n=h_{n-1}+n^3, \qquad h_0=0$.

Solution. Let $H(x):=\sum_{n=0}^{\infty} h_n\,x^n$. Then we have

$$\begin{aligned} H(x) &= h_0+\sum_{n=1}^{\infty} h_n\,x^n \\ &= \sum_{n=1}^{\infty} (h_{n-1}+n^3)\,x^n \\ &= \sum_{n=0}^{\infty} h_n\,x^{n+1}+\sum_{n=1}^{\infty} n^3\,x^n \\ &= x\,H(x)+\sum_{n=1}^{\infty} n^3\,x^n. \end{aligned} \tag{13}$$

To find $\sum_{n=1}^{\infty} n^3\,x^n$ we notice that

$$x\,\{x\,[x\,(x^n)']'\}'=n^3\,x^n. \tag{14}$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} n^3\,x^n &= \sum_{n=1}^{\infty} x\,\{x\,[x\,(x^n)']'\}' \\ &= x\left\{x\left[x\left(\sum_{n=0}^{\infty} x^n\right)'\right]'\right\}' \\ &= x\left\{x\left[x\left(\frac{1}{1-x}\right)'\right]'\right\}' \\ &= \frac{x}{(1-x)^2}+\frac{6\,x^2}{(1-x)^3}+\frac{6\,x^3}{(1-x)^4} \\ &= \frac{x+4\,x^2+x^3}{(1-x)^4}. \end{aligned} \tag{15}$$

Thus we have

$$H(x)=\frac{x+4\,x^2+x^3}{(1-x)^5}=(x+4\,x^2+x^3)\sum \frac{(n+4)\,(n+3)\,(n+2)\,(n+1)}{4!}\,x^n. \tag{16}$$

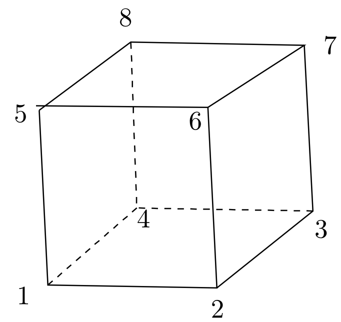
Thus the coefficient for x^n is

$$h_n=\frac{(n+3)\,(n+2)\,(n+1)\,n}{4!}+4\,\frac{(n+2)\,(n+1)\,n\,(n-1)}{4!}+\frac{(n+1)\,n\,(n-1)\,(n-2)}{4!}=\frac{n^2\,(n+1)^2}{4}. \tag{17}$$

More Examples

Example 3. How many paths are there from one corner of a cube to the opposite corner, each possible path being along three of the twelve edges of the cube?

Solution.



If we would like to reach 7 from 1 in 3 steps, then necessarily after the first step we have to be at one of 2,4,5; after the second step we have to be at one of 3,6,8. We have three choices in the first step, and after the first step we have two choices. After the second step we have only one choice. Therefore the total number of different paths is $3 \times 2 \times 1 = 6$.

Example 4. How many ways are there to distribute 18 toys to six children if each child receives a toy and the 18 toys can be divided into three groups of 6, 7, 5 each, and the toys within each group are identical?

Solution. The answer is:

$$|A_0| - |A_1 \cup \cdots \cup A_6| \tag{18}$$

where

$$A_0 = \{\text{ways distributing 18 toys to 6 children}\} \tag{19}$$

and

$$A_i = \{\text{The } i\text{th child receives no toy}\} \tag{20}$$

We see that

$$|A_0| = N_{01} \times N_{02} \times N_{03} \tag{21}$$

where N_{01} is the number of ways distributing 6 identical toys to 6 children, N_{02} the number of ways distributing 7 identical toys to 6 children, and N_{03} the number of ways distributing 5 identical toys to 6 children. Thus

$$|A_0| = \binom{11}{5} \times \binom{12}{5} \times \binom{10}{5}. \tag{22}$$

On the other hand, $|A_1 \cup \cdots \cup A_6|$ can be calculated through inclusion-exclusion. We have

$$|A_i| = \binom{10}{4} \times \binom{11}{4} \times \binom{9}{4}, \tag{23}$$

$$|A_i \cap A_j| = \binom{9}{3} \times \binom{10}{3} \times \binom{8}{3}, \tag{24}$$

$$|A_i \cap A_j \cap A_k| = \binom{8}{2} \times \binom{9}{2} \times \binom{7}{2}, \tag{25}$$

$$|A_i \cap A_j \cap A_k \cap A_l| = \binom{7}{1} \times \binom{8}{1} \times \binom{6}{1}, \tag{26}$$

$$|A_i \cap A_j \cap A_k \cap A_l \cap A_s| = 1. \tag{27}$$

Consequently

$$\begin{aligned}
\text{Ans} \;=\; & \binom{11}{5}\times\binom{12}{5}\times\binom{10}{5}- \\
& \binom{6}{1}\times\binom{10}{4}\times\binom{11}{4}\times\binom{9}{4}+ \\
& \binom{6}{2}\times\binom{9}{3}\times\binom{10}{3}\times\binom{8}{3}- \\
& \binom{6}{3}\times\binom{8}{2}\times\binom{9}{2}\times\binom{7}{2}+ \\
& \binom{6}{4}\times\binom{7}{1}\times\binom{8}{1}\times\binom{6}{1}- \\
& \binom{6}{5}\times 1.
\end{aligned} \tag{28}$$

Example 5. In how many ways can 25 different books be assigned to five different bookshelves if the order of the books on each shelf is considered important?

Solution. We mark the bookshelves 1,2,3,4,5. The reasonable assumption is that the number of books on each bookshelf is fixed, then putting the five bookshelves in a line (1–5 from left to right) we see that each distribution of the books corresponds to one permutation of the 25 books. So there are 25! different ways.

Remark. If the number of books on each bookshelf is not fixed, then for each single permutation of the 25 books there are $C(24,4)$ different distributions (if we do not allow empty bookshelves) or $C(29,4)$ different distributions (if we allow empty bookshelves). So the final answer would be

$$C(24,4)\times 25! \qquad \text{or} \qquad C(29,4)\times 25!. \tag{29}$$