Introduction

1. Block designs

Example 1. A consumer organization wishes to compare seven brands of detergent and arranges a number of tests. But since it may be uneconomic or inconvenient for each tester to compare all seven brands it is decided that each tester should compare just three brands. How should the trials be organized if each brand is to be tested the same number of times and each pair of brands is to be compared at least once directly by the same tester?

We see that one possible arrangement is, if we call the seven brands $A, B, C, ..., G$.

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \\ D & E & F & G & A & B & C \end{array}. \tag{1}$$

Here each tester compares just three brands, and each brand is tested by three testers. Moreover, any two brands are compared directly by one tester.

Exercise 1. Given these requirements, is it possible to have the job done with less than seven testers? (Hint:[1])

Exercise 2. Is it possible to have each tester test four brands, so that each brand is tested the same number of times and each pair of brands is tested exactly one time?

Example 2. A farmer wishes to compare eleven varieties of pig-feed and, again for reasons of economy or convenience, needs to arrange an experiment in which each trial involves just five varieties. How should the trials be organized if each pair of varieties is to be compared directly the same number of times?

One possible arrangement is as follows.

$$\begin{array}{ccccccccccc} A & A & A & A & A & B & B & B & C & C & D \\ B & B & C & D & E & C & D & E & E & F & G \\ C & F & D & F & H & D & E & G & F & H & I \\ G & I & E & G & J & H & F & H & G & I & J \\ J & K & I & H & K & K & J & I & K & J & K \end{array}. \tag{2}$$

Here each trial involves just five varieties, and each variety appears in five trials. Moreover, any two different varieties are directly compared in two trials.

Exercise 3. Show that to meet the requirements, we need at least eleven tests. Thus arrangement (2) is optimal.

2. Latin squares

Example 3. Consider testing the tread wear of four brands 1, 2, 3, 4, of tires. Naturally, four tires are put on a car and tested during each test, so each test involves one car and four tires. Clearly, to eliminate bias introduced by the different cars/drivers and the position of the tire on the car (front left/right, rear left/right), we would like to test all possible combinations of tire/position/car and thus should do more than

1. There are 21 pairs.

one tests. It turns out to be possible to achieve this with four cars.

	Car A	Car B	Car C	Car D
Left front	1	2	3	4
Right front	2	3	4	1
Left rear	3	4	1	2
Right reat	4	1	2	3

(3)

Example 4. Suppose that we wish to test four types of fertilizer on a field of wheat. A simple method for comparing them might be to divide the field into four parallel strips and test one type of fertilizer on each strip. But this could give inaccurate results if a row of trees gave shade along one side of the field or there was a waste tip in the middle. Such inaccuracies may be lessened by dividing the field into small squares, with one fertilizer applied to each square of field, such that the squares with the same fertilizer located at different rows and columns. For example,

	Column 1	Column 2	Column 3	Column 4
Row 1	1	2	3	4
Row 2	2	3	4	1
Row 3	3	4	1	2
Row 4	4	1	2	3

(4)

THEORY OF BLOCK DESIGNS

DEFINITION 5. (BLOCK DESIGN) *A block design consists of a set of v varieties arranged into b blocks, such that each block contains the same number k of varieties and each variety appears in the same number r of blocks. Such a arrangement is called a (v, b, r, k) design or (v, k) design.*

Remark 6. For example, if we would like to test v drugs using b tests, each testing k drugs, and sucht hat each drug is tested r times, then we need a (v, b, r, k) design.

Example 7. In the following block design we have $v = 9, b = 6, k = 3, r = 2$:

$$\begin{array}{cccccc} A & A & B & C & D & E \\ B & E & F & D & C & F \\ G & G & I & H & H & I \end{array}. \tag{5}$$

LEMMA 8. *In every block design we have $v\,r = b\,k$.*

Proof. Clearly $b\,k$ and $v\,r$ are both the total number of symbols in the array. Therefore they equal. $\square$

DEFINITION 9. (BALANCED BLOCK DESIGN) *A block design is called "balanced" if any pair of two varieties appear in the same number of blocks. This number is usually denoted λ. We denote such design as (v, b, r, k, λ)-design or (thanks to Lemma 8) (v, k, λ)-design.*

Example 10. (1), (2) are balanced, but (5) is not.

LEMMA 11. *In every balanced block design we have $r\,(k-1) = \lambda\,(v-1)$.*

Proof. As there are v varieties, there are $\binom{v}{2}$ pairs of varieties. Each pair is tested λ times. Therefore the total number of tests for pairs is

$$\lambda \binom{v}{2} = \frac{\lambda\,v\,(v-1)}{2}. \tag{6}$$

On the other hand, each block has k varieties and thus each test involves $\binom{k}{2}$ tests of pairs. Consequently the total number of tests for pairs of

$$b \binom{k}{2} = \frac{b\,k\,(k-1)}{2}. \tag{7}$$

By Lemma 8 we have $b\,k = v\,r$ which leads to

$$\frac{\lambda\,v\,(v-1)}{2} = \frac{r\,v\,(k-1)}{2} \Longrightarrow \lambda\,(v-1) = r\,(k-1). \tag{8}$$

Thus ends the proof. $\square$

PROPOSITION 12. (FISHER'S INEQUALITY) *A block design is called "incomplete" if $k < v$. In a balanced incomplete block design (BIBD) we have $v \leqslant b$.*

Proof. Let A be the incidence matrix of the design, that is A is a $v \times b$ matrix of 0's and 1's such that $a_{ij} = 1$ if and only if the ith symbol is in the jth block.

We claim that

$$A\,A^T = \begin{pmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{pmatrix}. \tag{9}$$

Exercise 4. Prove this.

Now if $b < v$, then $\det(A\,A^T) = 0$ since $A\,A^T$ is a $v \times v$ matrix with rank at most b. Now subtract the first column from every other column we see that

$$0 = \det\begin{pmatrix} r & \lambda - r & \lambda - r & \cdots & \lambda - r \\ \lambda & r - \lambda & 0 & \cdots & 0 \\ \lambda & 0 & r - \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & \cdots & r - \lambda \end{pmatrix}. \tag{10}$$

Now add all the other rows to the first row we have

$$0 = \det\begin{pmatrix} r + (v-1)\,\lambda & 0 & 0 & \cdots & 0 \\ \lambda & r - \lambda & 0 & \cdots & 0 \\ \lambda & 0 & r - \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & \cdots & r - \lambda \end{pmatrix} \tag{11}$$

which means

$$[r + (v-1)\,\lambda]\,(r - \lambda)^{v-1} = 0 \Longrightarrow r = \lambda. \tag{12}$$

By Lemma 11 this implies $k = v$. Contradiction. $\square$

Remark 13. A balanced design with $v = b$ is called symmetric.

Exercise 5. Prove that in a symmetric design there holds $k = r$ and consequently $\lambda\,(v-1) = k\,(k-1)$.