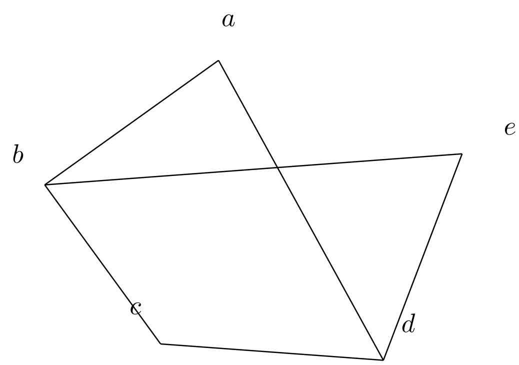
Definition 1. (Simple Graph) *A simple graph is a pair of two finite sets: A set of "vertices" $V = \{a, b, c, ...\}$, and a set of "edges" $E$ whose elements are subsets of size 2 of the set $V$. We denote the graph by $G = (V, E)$.*

**Example 2.** Consider the graph $G = (V, E)$ where

$$V = \{a, b, c, d, e\}, \qquad E = \{\{a, b\}, \{a, d\}, \{b, e\}, \{e, d\}, \{b, c\}, \{c, d\}\}. \tag{1}$$

It is possible to visualize $G$ as follows.




It is important to realize that the drawing is just a visualization.

**Remark 3.** There are drawings of vertices-edges that are not simple graphs. For example when there are two different edges connecting the same two vertices, or when there is an edge connecting a vertex to itself. Those are called multigraphs or simply graphs. We will not discuss those more general graphs.

Definition 4. (Order) *Let $G = (V, E)$ be a simple graph. Then $n := |V|$ is called the "order" of the graph.*

Proposition 5. *Let $G$ be a simple graph of order $n$. Then $|E| \leqslant \frac{n(n-1)}{2}$.*

**Proof.** As $E$ consists of subsets of size 2 of $G$, and the total number of such subsets is given by $\binom{n}{2} = \frac{n(n-1)}{2}$, the conclusion follows. □

**Exercise 1.** Where did we need the hypothesis that $G$ is a simple graph?

**Example 6.** (Complete graph) A graph of order $n$ with $|E| = \frac{n(n-1)}{2}$ is called the "complete graph of order $n$", denoted $K_n$.

Definition 7. (Degree) *The* degree *of a vertex $x$ in a graph $G$ is the number of edges that are connected to $x$. We denote it by $\deg(x)$. To each graph, we can associate a sequence of numbers that is the list of the degrees of the graph's vertices in nonincreasing order.*

$$(d_1, ..., d_n), \qquad d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 0. \tag{2}$$

*This list is called the* degree sequence *of the graph $G$.*

**Example 8.** The degree sequence for $K_n$ is $(n-1, n-1, ..., n-1)$.

Theorem 9. (Euler) *Let $G$ be a graph. Then $d_1 + \cdots + d_n$ is even.*

**Proof.** We see that, when we add all the degrees, we are counting each edge exactly twice. Therefore

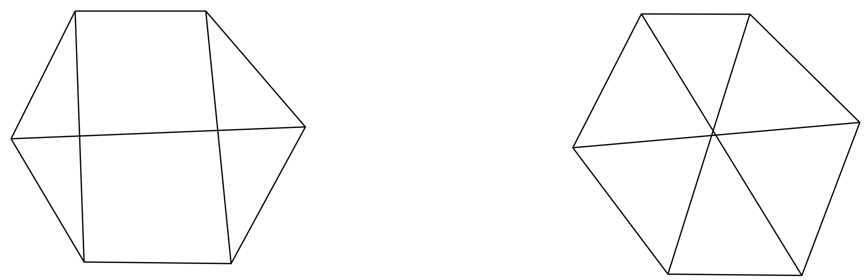
$$d_1 + \cdots + d_n = 2|E| \tag{3}$$

and is therefore even. □

**Exercise 2.** At a party, a lot of handshaking takes place between the guests. Show that, at the end of the party, the number of guests who have shaken hands an odd number of times is even.

**Exercise 3.** Prove that two isomorphic graphs have the same degree sequence.

**Example 10.** It turns out that the degree sequence does not determine the graph. Consider the following two graphs.



We see that both graphs have degree sequence $(3,3,3,3,3,3)$. However the two graphs are not isomorphic, as can be seen from the fact that the left graph contains a triangle while the right one does not.

**Exercise 4.** Prove that every simple graph has at least two vertices of the same degree.

**Example 11.** Mr. and Mrs. Smith invited four couples to their home. Some guests were friends of Mr. Smith, and some others were friends of Mrs. Smith. When the guests arrived, people who knew each other beforehand shook hands, those who did not know each other just greeted each other. After all this took place, the observant Mr. Smith said "How interesting. If you disregard me, there are no two people present who shook hands the same number of times".

How many times did Mrs. Smith shake hands?

**Remark.** There is a hidden assumption that nobody shook hands with himself/herself or his/her spouse.

**Solution.** Let's denote the other four couples by $\{a,\alpha\}$, $\{b,\beta\}$, $\{c,\gamma\}$, $\{d,\delta\}$. The handshaking situation can be represented as a graph with ten vertices $\{a,\alpha,b,\beta,c,\gamma,d,\delta,\mathrm{Mrs},\mathrm{Mr}\}$ and two vertices are connected by an edge when they shake hands.

According to Mr. Smith's remark, the degrees at $a$, $\alpha$, ..., Mrs.Smith are nine different numbers from 0, 1, 2, ..., 8, 9. But if a vertex has degree 9, then it is connected to every other vertex. This means the corresponding person shook hands with his/her spouse too. Therefore the degrees of the other nine vertices are $0,1,2,...,8$.

First note that $\deg(\mathrm{Mrs}) \neq 8$. As if $\deg(\mathrm{Mrs}) = 8$, since it is nor connected to Mr, it must be connected to the remaining eight vertices, and none of these vertices can have degree 0 anymore.

Without loss of generality, assume $\deg(a) = 8$. Then $a$ is connected to eight other vertices and there is exactly one vertex not connected to $a$. This vertex must be $\alpha$. Consequently $\deg(\alpha) = 0$.

Now we claim that $\deg(\mathrm{Mrs}) \neq 7$. As if $\deg(\mathrm{Mrs}) = 7$, since $\deg(\alpha) = 0$ and Mrs is not connected to Mr, there must hold that Mrs is connected to all the remaining seven vertices. As a consequence, none of these vertices can have degree 1.

Without loss of generality, assume $\deg(b) = 7$. Then a similar argument as above shows that $\deg(\beta) = 1$.

Similarly, we conclude $\deg(c) = 6, \deg(\gamma) = 2$ and $\deg(d) = 5, \deg(\delta) = 3$.

As a consequence, $\deg(\mathrm{Mrs}) = 4$, that is Mrs. Smith shook hands 4 times.

**Exercise 5.** How many times did Mr. Smith shake hands?

**Exercise 6.** What if there are $n$ couples invited?

**Exercise 7.** There are ten people at a party. Some shook hands and some did not. Assume that nobody shook hands with the same person twice or shook hands with him/herself. At the end of the party one person make the remark that "if you disregard me, there are no two people present who shook hands the same number of times." Can we determine how many times each person shook hands?

**Example 12.** (Ramsey Theory) In any party of six or more people, there are either at least three mutual friends (who all know each other) or at least three mutual non-friends (who know neither of the other two).

**Proof.** It is clear that we only need to prove this for exactly six people. We represent each person by a vertex. Two vertices are connected by an edge if and only if they know each other. Thus the problem becomes, in a simple graph of order 6, there exists three vertices such that either they are connected into a triangle, or there is no edge between any two of them.

Denote the six vertices by $a, b, c, d, e, f$. Consider the vertex $a$. There are two cases:

i. $\deg(a) \geqslant 3$. Without loss of generality, assume that $a$ is connected to $b, c, d$. Now there are four cases.

   a) $\{b, c\} \in E$. Then $a, b, c$ form a triangle.

   b) $\{c, d\} \in E$. Then $a, c, d$ form a triangle.

   c) $\{d, b\} \in E$. Then $a, d, b$ form a triangle.

   d) None of $\{b, c\}, \{c, d\}, \{d, b\} \in E$. Then the three vertices $b, c, d$ have no edge between any two of them.

ii. $\deg(a) \leqslant 2$. We leave this situation as an exercise. □

**Exercise 8.** Show that six is the smallest number having this property (either at least three mutual friends or at least three mutual non-friends).

**Remark 13.** It can be show that to have four mutual friends or mutual non-friends we need at least eighteen people, and to have five mutual friends or mutual non-friends we need somewhere between forty three to forty nine people.