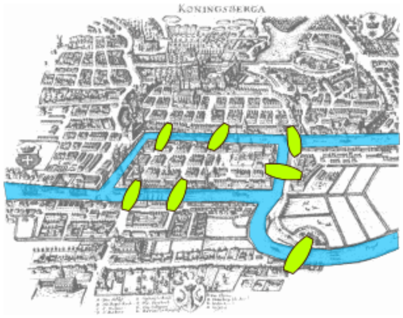
A graph consists of a number of points, called vertices, and lines joining them in pairs, called edges; the edges can be drawn straight or curved.

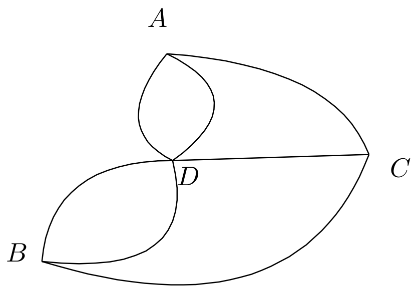
## Examples

**Example 1. (the Seven bridges of Koenigsberg)** Graph theory started from the following problem.



Question: Is it possible to walk through the city crossing each of the seven bridges exactly once?

**Solution.** We first simplify the situation as follows.




We see that the graph representation involves four vertices and seven edges. The question is whether it is possible to have a path containing the seven edges without repeating any of them. A moment's reflection reveals that any such path must start from a vertex with three edges and end at another vertex with three edges, and any passing vertex can only have an even number of edges. As A,B,C,D all have three edges emanating from them, it is not possible to walk through the city crossing each of the seven bridges exactly once.

**Example 2. (The four color problem)** Is it always possible to use four colors to color any map drawn in the plane so that any two countries having a common border have different colors?

Naively we can turn this problem into the following graph coloring problem. We identify each country as a vertex, and put an edge connecting two vertices when the two countries are neighbours. Then the question becomes: Is it always possible to color the vertices of a graph with four colors so that any two vertices connected by an edge are colored differently?

**Exercise 1.** Show that the answer is no.

The above identification (coloring a map $\Longleftrightarrow$ coloring a graph) is naïve because graphs representing maps are special: it is possible to draw the graph in a plane with no edges crossing. Such graphs are called "planar". It is a nontrivial problem to determine whether a graph is planar or not.

Thus the correct graph theoretic question is: Is it always possible to use four colors to color the vertices of any planar graph?

**Exercise 2.** Show that if we replace "four" by "three" then the answer is no.

**Exercise 3.** Show that the answer to the original map-coloring problem is no if we allow one country to consist of more than one pieces.

**Remark 3.** In a later section we will prove the claim when “four” is replaced by “five”.[1]

**Proposition 4.** *If $k$ colors are enough to color a plane map, then $k$ colors are also enough to color a spherical map.*

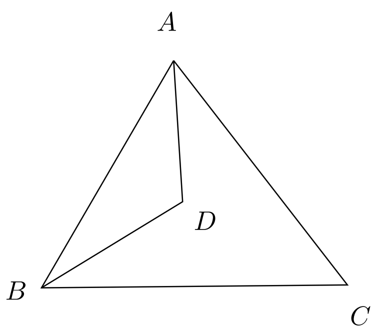
**Proof.** We pick an arbitrary point that is within a country, and make it the north pole. Application of the stereographic projection (https://en.wikipedia.org/wiki/Stereographic_projection) now turns the spherical map to a planar map and the conclusion readily follows. □

**Remark 5.** We will see that this is not true for surfaces that are “truly different” from spheres.

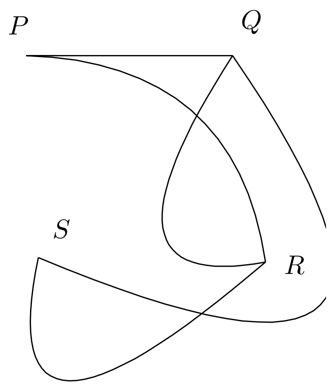
**Example 6. (Graph coloring)** In general, graph coloring problems often arise from scheduling. For example, consider a university with $n$ courses. To schedule the final exams for these courses, one has to make sure that exams for courses that share students are not scheduled at the same time. The task now is to minimize the time slots needed.

To turn this into a graph coloring problem, we use $n$ vertices to represent the courses and draw an edge between two vertices exactly when the two corresponding courses share one or more students. Now we ask: What is the smallest number of colors needed to color the graph so that any two vertices that are connected by an edge are colored differently?

**Example 7. (Graph isomorphism; How many ways to color a graph)** In graph theory, there are many different ways to draw the same graph. For example




and




are the same graph, through the identification

$$A \longleftrightarrow Q, \quad B \longleftrightarrow R, \quad C \longleftrightarrow P, \quad D \longleftrightarrow S. \tag{1}$$

It is an important problem to determine whether two graphs are the “same”, or isomorphic. A correspondence like (1) is called a graph isomorphism.

**Exercise 4.** Show that, the collection of graph isomorphisms from one graph to itself form a group.

Another important question is to ask, for example, how many different ways are there to color the vertices of a graph. Here, if one coloring can be obtained from another through a graph automorphism, then the two colorings are consider the same.

1. This is proved in P. J. Heawood, *Map-color theorems*, Quarterly J. Mathematics, Oxford ser. 24 (1890), 332–338.

**Exercise 5.** How many ways are there to color the vertices $A, B, C, D$ with three colors?