# Generating Functions

- Generating function is a method to systematically study a sequence of numbers $a_0, a_1, \ldots$ . Here $a_0$, $a_1, \ldots, a_n, \ldots$ are answers to combinatoric problem with a parameter $n$ involved. There are two popular types of generating functions,
  - the ordinary generating function

    $$A(x) := a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

    and
  - the exponential generating function

    $$E(x) := a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n. \tag{2}$$

  - For occupancy problems, identical balls $\Longrightarrow$ use ordinary generating function, different balls $\Longrightarrow$ use exponential generating function.
- Once we have the generating function, the numbers $a_n$ can be obtained through Taylor expansion of the function.
  - For ordinary generating functions, often we need to do Taylor expansion of a rational function. This is done through
    i. Partial fraction;
    ii. The (most important) expansion

    $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{3}$$

    and the expansions for $\frac{1}{(1-x)^k}$ obtained from differentiating (3).
- To determine the generating function of a problem, one can
  - try to write down the generating function directly, or
  - try to solve the generating function from a recurrence relation of the $a_n$'s.
- Examples.

**Example 1.** [1]Starting with a set of $n$ elements $\{a, b, c, \ldots\}$, consider the set of combinations, with repetition allowed, where each element appears an even number of times. For example *aaccdddd* is a legal combination, but *aabbb* is not. Let the number of such combinations with $k$ elements be $a_k$. Find $a_k$ through generating functions if

a) Order does not matter (that is *aabbb* is the same as *abbab*.)

b) Order matters (that is *aabbb* is not the same as *abbab* while both are legal.)

**Solution.**

a) We see that the problem is equivalent to putting $k$ identical balls into $n$ distinct boxes, thus $a_k$ is the number of solutions to

$$x_1 + \cdots + x_n = k \tag{4}$$

1. http://www.math.ualberta.ca/~isaac/math421/w06/mt_review.pdf.

with the extra requirement that each $x_i$ is even. Therefore the generating function is

$$\begin{aligned} A(x) = (1 + x^2 + x^4 + \cdots)^n &= \frac{1}{(1-x^2)^n} \\ &= \frac{1}{(n-1)!} \sum_{l=0}^{\infty} (l+n-1)\cdots(l+1)\, x^{2l} \\ &= \binom{l+n-1}{l} x^{2l}. \end{aligned} \tag{5}$$

Therefore we have $a_k = \binom{l+n-1}{l}$ when $k$ is even and $a_k = 0$ when $k$ is odd.

b) In this case the problem is equivalent to putting $k$ distinct balls (the $k$ positions) into $n$ distinct boxes (the $n$ symbols), with the further requirement that each box contains an even number of balls. Thus we use exponential generating function:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k = E(x) &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^n \\ &= \left(\frac{e^x + e^{-x}}{2}\right)^n \\ &= 2^{-n} \sum_{l=0}^{n} \binom{n}{l} e^{lx}\, e^{(n-l)(-x)} \\ &= 2^{-n} \sum_{l=0}^{n} \binom{n}{l} e^{(2l-n)x} \\ &= 2^{-n} \sum_{k=0}^{\infty} \frac{\sum_{l=0}^{n} \binom{n}{l} (2\,l-n)^k}{k!} x^k. \end{aligned} \tag{6}$$

Therefore

$$a_k = \frac{\sum_{l=0}^{n} \binom{n}{l} (2\,l-n)^k}{2^n}. \tag{7}$$

**Example 2.** [2]Find the ordinary generating function for the sequence $\{a_n\}_{n \geqslant 0}$ satisfying

$$a_n = 2\,a_{n-1} + 1, \qquad n \geqslant 1, \qquad a_0 = 0 \tag{8}$$

and use it to find $a_n$.

**Solution.** We have

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n\, x^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n\, x^n \\ &= 0 + \sum_{n=1}^{\infty} (2\,a_{n-1} + 1)\, x^n \\ &= \sum_{n=1}^{\infty} 2\,a_{n-1}\, x^n + \sum_{n=1}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} 2\,a_n\, x^{n+1} + \frac{x}{1-x} \\ &= 2\,x\,A(x) + \frac{x}{1-x} \Longrightarrow A(x) = \frac{x}{(1-x)\,(1-2\,x)}. \end{aligned} \tag{9}$$

Now apply partial fraction:

$$\frac{x}{(1-x)\,(1-2\,x)} = \frac{A}{1-x} + \frac{B}{1-2\,x} \Longrightarrow x = A\,(1-2\,x) + B\,(1-x) \tag{10}$$

which gives

$$A = -1, \qquad B = 1. \tag{11}$$

2. http://www.math.ualberta.ca/~isaac/math421/w06/fn_review.pdf.

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= A(x)\\ &= \frac{1}{1-2x} - \frac{1}{1-x}\\ &= \sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n\\ &= \sum_{n=0}^{\infty} (2^n - 1)\, x^n. \qquad (12)\end{aligned}$$

Thus $a_n = 2^n - 1$.

**Example 3.** Find and solve a recurrence relation for the number of ways to make a pile of $n$ poker chips using red, white, and blue chips and such that no two red chips are together.

**Solution.** We have clearly $a_1 = 3$ and $a_2 = 8$. Now let $n > 2$. Consider the color of the top chip. If it is red, then the one below it cannot be red and the remaining $n-2$ chips give $a_{n-2}$ different ways. If it is not red, then the remaining $n-1$ chips give $a_{n-1}$ different ways. Therefore the recurrence relation is

$$a_n = 2\, a_{n-1} + 2\, a_{n-2}, \qquad n > 2, \qquad a_1 = 3, a_2 = 8. \qquad (13)$$

We have

$$\begin{aligned}A(x) &:= \sum_{n=1}^{\infty} a_n x^n\\ &= 3x + 8x^2 + \sum_{n=3}^{\infty} a_n x^n\\ &= 3x + 8x^2 + \sum_{n=3}^{\infty} (2\, a_{n-1} + 2\, a_{n-2})\, x^n\\ &= 3x + 8x^2 + 2\sum_{n=2}^{\infty} a_n x^{n+1} + 2\sum_{n=1}^{\infty} a_n x^{n+2}\\ &= 3x + 8x^2 + 2\sum_{n=1}^{\infty} a_n x^{n+1} - 6x^2 + 2x^2 A(x)\\ &= 3x + 2x^2 + 2x\, A(x) + 2x^2 A(x) \Longrightarrow A(x) = \frac{3x + 2x^2}{1 - 2x - 2x^2}. \qquad (14)\end{aligned}$$

Solving this using partial fraction gives

$$a_n = \frac{1}{4\sqrt{3}} \left[ \left(1 + \sqrt{3}\right)^{n+2} - \left(1 - \sqrt{3}\right)^{n+2} \right]. \qquad (15)$$

**Example 4.** [3]Given the sequence $\{a_n\}_{n \geqslant 1}$ defined recursively by

$$a_{n+2} = 14\, a_{n+1} - a_n - 4, \qquad n \geqslant 1, \qquad a_1 = 1, a_2 = 1. \qquad (16)$$

Show that $a_n$ is a perfect square for all $n \geqslant 1$.

**Proof.** We have

$$\begin{aligned}A(x) &:= \sum_{n=1}^{\infty} a_n x^n\\ &= x + x^2 + \sum_{n=3}^{\infty} a_n x^n\\ &= x + x^2 + \sum_{n=3}^{\infty} (14\, a_{n-1} - a_{n-2} - 4)\, x^n\end{aligned}$$

3. http://www.math.ualberta.ca/~isaac/math421/w06/fn_review.pdf.

$$\begin{aligned}
&= x + x^2 + \sum_{n=3}^{\infty} 14\, a_{n-1}\, x^n - \sum_{n=3}^{\infty} a_{n-2}\, x^n - 4 \sum_{n=3}^{\infty} x^n \\
&= x + x^2 + 14 \sum_{n=2}^{\infty} a_n\, x^{n+1} - \sum_{n=1}^{\infty} a_n\, x^{n+2} - 4\, x^3 \sum_{n=0}^{\infty} x^n \\
&= x + x^2 - 14\, x^2 + 14 \sum_{n=1}^{\infty} a_n\, x^{n+1} - x^2\, A(x) - \frac{4\, x^3}{1-x} \\
&= x - 13\, x^2 + 14\, x\, A(x) - x^2\, A(x) - \frac{4\, x^3}{1-x}. \qquad (17)
\end{aligned}$$

Therefore

$$A(x) = \frac{x - 13\, x^2}{1 - 14\, x + x^2} - \frac{4\, x^3}{(1 - 14\, x + x^2)\,(1-x)}. \qquad (18)$$

This gives

$$A(x) = x\, \frac{(1 - 13\, x)\,(1-x) - 4\, x^2}{(1 - 14\, x + x^2)\,(1-x)} = x\, \frac{1 - 14\, x + 9\, x^2}{(1 - 14\, x + x^2)\,(1-x)} = \frac{x}{1-x} + x\, \frac{8\, x^2}{(1 - 14\, x + x^2)\,(1-x)}. \qquad (19)$$

We apply partial fraction expansion to the latter term. First note that

$$1 - 14\, x + x^2 = \left[\left(7 + 4\sqrt{3}\right) - x\right]\left[\left(7 - 4\sqrt{3}\right) - x\right] \qquad (20)$$

so

$$\frac{8\, x^2}{(1 - 14\, x + x^2)\,(1-x)} = \frac{A}{\left(7 + 4\sqrt{3}\right) - x} + \frac{B}{\left(7 - 4\sqrt{3}\right) - x} + \frac{C}{1-x}. \qquad (21)$$

Solving this gives

$$A = \frac{\left(7 + 4\sqrt{3}\right)^2}{12 + 6\sqrt{3}}, \qquad B = \frac{\left(7 - 4\sqrt{3}\right)^2}{12 - 6\sqrt{3}}, \qquad C = -\frac{2}{3}. \qquad (22)$$

Now notice that

$$\left(2 \pm \sqrt{3}\right)^2 = 7 \pm 4\sqrt{3}. \qquad (23)$$

Consequently

$$A = \frac{1}{6}\left(2 + \sqrt{3}\right)^3, \qquad B = \frac{1}{6}\left(2 - \sqrt{3}\right)^3, \qquad C = -\frac{2}{3}. \qquad (24)$$

Therefore

$$\begin{aligned}
A(x) &= \frac{1}{3}\,\frac{x}{1-x} + \frac{1}{6}\left[\frac{\left(2+\sqrt{3}\right)^3 x}{\left(2+\sqrt{3}\right)^2 - x} + \frac{\left(2-\sqrt{3}\right)^3 x}{\left(2-\sqrt{3}\right)^2 - x}\right] \\
&= \frac{1}{3}\,\frac{x}{1-x} + \frac{1}{6}\left[\frac{\left(2+\sqrt{3}\right) x}{1 - \left(2-\sqrt{3}\right)^2 x} + \frac{\left(2-\sqrt{3}\right) x}{1 - \left(2+\sqrt{3}\right)^2 x}\right] \\
&= \frac{x}{3} \sum_{n=0}^{\infty} x^n + \frac{x}{6} \sum_{n=0}^{\infty} \left[\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)^{2n} + \left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2n}\right] x^n \\
&= \frac{1}{3} \sum_{n=1}^{\infty} x^n + \frac{1}{6} \sum_{n=1}^{\infty} \left[\left(2-\sqrt{3}\right)^{2n-3} + \left(2+\sqrt{3}\right)^{2n-3}\right] x^n. \qquad (25)
\end{aligned}$$

Therefore

$$\begin{aligned}
a_n &= \frac{1}{6}\left[\left(2-\sqrt{3}\right)^{2n-3} + 2 + \left(2+\sqrt{3}\right)^{2n-3}\right] \\
&= \frac{1}{36}\left[\left(12 - 6\sqrt{3}\right)\left(2-\sqrt{3}\right)^{2n-2} + 2 + \left(12 + 6\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2n-2}\right] \\
&= \frac{1}{36}\left[\left(3-\sqrt{3}\right)^2\left(2-\sqrt{3}\right)^{2n-2} + 2 + \left(3+\sqrt{3}\right)^2\left(2+\sqrt{3}\right)^{2n-2}\right] \\
&= \left[\frac{\left(3-\sqrt{3}\right)\left(2-\sqrt{3}\right)^{n-1} + \left(3+\sqrt{3}\right)\left(2+\sqrt{3}\right)^{n-1}}{6}\right]^2. \qquad (26)
\end{aligned}$$

Through binomial expansion we see that the number getting squared is rational (all the $\sqrt{3}$ terms cancel). A rational number square to an integer if and only if it is itself an integer. Therefore $a_n$ is a perfect square. □