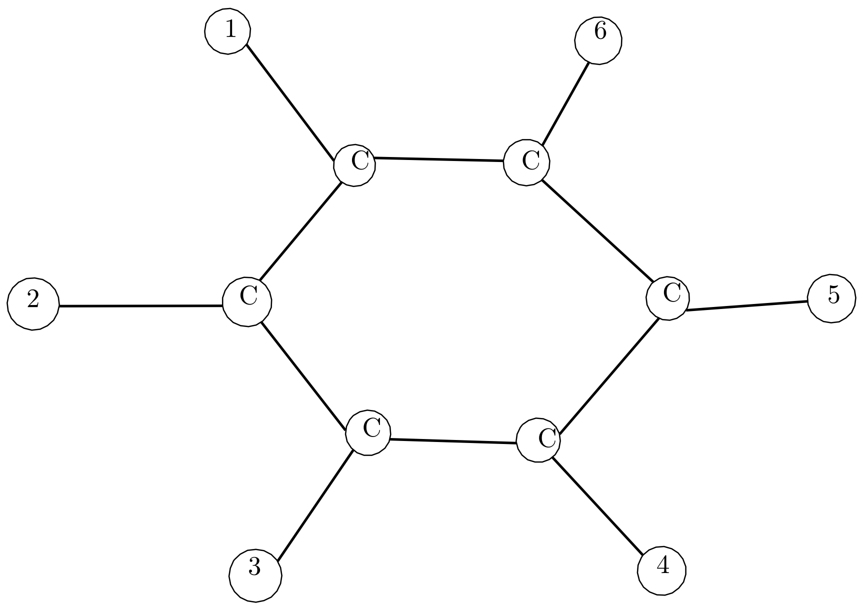
Example 1. In a benzene ring, six carbon atoms are attached to each other in the shape of a regular hexagon. If two kinds of radicals can be attached to each carbon atom and all the C-C bonds in the ring are assumed to be equivalent, how many different chmical compounds can be formed?



Solution. This is the same as coloring the vertices of a regular hexagon with two colors. We see that the symmetry group G has 12 elements.

- Identity: $i = (1)(2)(3)(4)(5)(6)$. We have $c(i) = 6$;
- Rotation of 60: $g = (123456)$. We have $c(g) = 1$;
- Rotation of 120 degrees: $g = (135)(246)$. We have $c(g) = 2$;
- Rotation of 180 degrees: $g = (14)(25)(36)$. We have $c(g) = 3$;
- Rotation of 240 degrees: Same as 120 degrees;
- Rotation of 300 degrees: Same as 60 degrees;
- Flipping around axis 1-4: $g = (1)(4)(26)(35)$. We have $c(g) = 4$.
- Flipping around axis 2-5 or 3-6: Same as around 1-4.
- Flipping around axis connecting the middle points of 12 and 45: $g = (12)(36)(45)$. $c(g) = 3$.
- The other two flippings: Same.

Therefore the answer is

$$\frac{2^6 + 2 \times 2^1 + 2 \times 2^2 + 4 \times 2^3 + 3 \times 2^4}{12} = 13. \tag{1}$$

Exercise 1. In how many ways can a $2 \times 2 \times 2$ cube be constructed from eight $1{\times}1{\times}1$ cubes if an unlimited number of red, white, and blue cubes are available?

Example 2. Consider three identical balls and three identical rods connected into an equilateral triangle, with the three balls at the vertices. We would like to color the balls with two colors and the the rods with three colors. How many different ways are there to do this?

Solution. Let X_b be the possible ways of coloring three marked balls with two colors, and let X_r be the possible ways of coloring three marked rods with three colors. Then

$$X = X_b \times X_r := \{(x_b, x_r) | \, x_b \in X_b, x_r \in X_r\}. \tag{2}$$

Now the key observation is that, there is only one symmetry group: the symmetry group of the equilateral triangle. G acts on x through

$$g\,x := (g\,x_b, g\,x_r). \tag{3}$$

for $x = (x_b, x_r) \in X$, we have

$$\mathrm{Orb}(x) = \mathrm{Orb}(x_b) \times \mathrm{Orb}(x_r) := \{(g\,x_b, g\,x_r) |\, g \in G\}, \tag{4}$$

and

$$X_g = \{x \in X |\, g\,x = x\} = \{(x_b, x_r) |\, g\,x_b = x_b, g\,x_r = x_r\} = X_{b,g} \times X_{r,g}. \tag{5}$$

Consequently, there holds

$$|X_g| = |X_{b,g}|\, |X_{r,g}|. \tag{6}$$

Now if we apply Polya's theory, we obtain

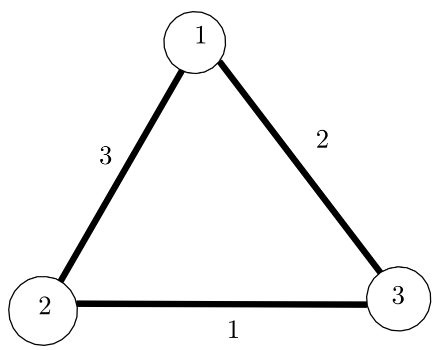
$$|X_g| = 2^{c_b(g)}\, 3^{c_r(g)}, \tag{7}$$

where $c_b(g)$ is the number of cycles in the cyclic representation of g when we view G as a subgroup of the permutation group of the marked balls, and $c_r(g)$ is the number of cycles in the cyclic representation of g when we view G as a subgroup of the permutation group of the marked rods.

Therefore the answer is given by

$$\mathrm{Ans} = \frac{1}{|G|} \sum_{g \in G} 2^{c_b(g)}\, 3^{c_r(g)}. \tag{8}$$

Now we are ready to solve the problem. We mark the balls and rods as follows:



We recall that the symmetry group of the equilateral triangle:

- i: Identity. For the balls we have $i = (1)(2)(3)$, for the rods we also have $i = (1)(2)(3)$. Thus $c_b(i) = c_r(i) = 3$.
- r_1: Counter-clockwise rotation of 120 degrees. For both the balls and rods we have $r_1 = (123)$. Thus $c_b(r_1) = c_r(r_1) = 1$.
- r_2: Counter-clockwise rotation of 240 degrees. Similarly we have $c_b(r_2) = c_r(r_2) = 1$.
- f_1: Flipping with respect to the line passing ball 1 and the middle of rod 1. For both the balls and rods we hvae $f_1 = (1)(23)$ so $c_b(f_1) = c_r(f_1) = 2$.
- f_2, f_3: Same as f_1.

Consequently the answer is

$$\frac{2^3\, 3^3 + 2 \times 2^1\, 3^1 + 3 \times 2^2\, 3^2}{6} = 56. \tag{9}$$

Remark 3. In the above example we have $c_b(g) = c_r(g)$ for all $g \in G$. In general this is not true.

Example 4. Consider a cube with a bead at each vertex. We would like to use two colors R and G to color both the faces and the beads at the vertex. How many different ways are there?

Solution. Again, for every $g \in G$ the symmetry group of the cube,

$$X_g = X_g^f \times X_g^v \Longrightarrow |X_g| = \left|X_g^f\right| \, |X_g^v|. \tag{10}$$

Thus we have (Here we use the shorthand: $(1)^2(4)^1$ means there are two 1-cycles and one 4-cycle.

$g \in G$	Faces	Vertices
i	$(1)^6$	$(1)^8$
3 rotation by $\pi/2$ around the line connecting face centers	$(1)^2\,(4)^1$	$(4)^2$
3 rotation by π around the line connecting face centers	$(1)^2\,(2)^2$	$(2)^4$
3 rotation by $3\pi/2$ around the line connecting face centers	$(1)^2\,(4)^1$	$(4)^2$
6 rotations by π around the line connecting middle points of oppositing edges	$(2)^3$	$(2)^4$
4 rotations by $2\pi/3$ around long diagonals	$(3)^2$	$(1)^2\,(3)^2$
4 rotations by $4\pi/3$ around long diagonals	$(3)^2$	$(1)^2\,(3)^2$

Finally we see that the answer is given by

$$\frac{2^{6+8} + 3 \times 2^{2+1+2} + 3 \times 2^{2+2+4} + 3 \times 2^{2+1+2} + 6 \times 2^{3+4} + 4 \times 2^{2+2+2} + 4 \times 2^{2+2+2}}{24} = 776. \tag{11}$$

Exercise 2. How many ways are there to color a regular tetrahedron using four colors for vertices, three colors for faces, and two colors for edges? (Ans:[1])

Example 5. How many ways to color the six vertices of a regular hexagon with four colors, such that two vertices are red, two are blue, one is green, and one is yellow?

Solution. It turns our that we can "track" the colors used through the following trick: For each k-cycle, instead of just 4, representing there are four possible ways to color this cycle, we write it as $(r^k + g^k + b^k + y^k)$, representing the four possible ways: all k vertices red, all k vertices green, all k vertices blue, all k vertices yellow.

Now we go through all 12 elements of the symmetry group for a regular hexagon.

- $i = (1)(2)(3)(4)(5)(6)$. The contribution is $(r + g + b + y)^6$.
- $r_1 = (123456)$. The contribution is $(r^6 + g^6 + b^6 + y^6)$.
- $r_2 = (135)(246)$ yields $(r^3 + g^3 + b^3 + y^3)^2$.
- $r_3 = (14)(25)(36)$ yields $(r^2 + g^2 + b^2 + y^2)^3$.
- $r_4 = (153)(264)$ yields $(r^3 + g^3 + b^3 + y^3)^2$.
- $r_5 = (165432)$ yields $(r^6 + g^6 + b^6 + y^6)$.
- $f_1 = (16)(25)(34)$ yields $(r^2 + g^2 + b^2 + y^2)^3$. Same for f_2, f_3.
- $f_4 = (1)(4)(26)(35)$ yields $(r + g + b + y)^2\,(r^2 + g^2 + b^2 + y^2)^2$. Same for f_5, f_6.

Thus all possible colorings are encoded in the following formula, which in a sense is a "generating function" for our coloring problem.

$$\frac{1}{12}\left[(r+g+b+y)^6 + 2\,(r^6+g^6+b^6+y^6) + 2\,(r^3+g^3+b^3+y^3)^2 + 4\,(r^2+g^2+b^2+y^2)^3 + 3\,(r+g+b+y)^2\,(r^2+g^2+b^2+y^2)^2\right] \tag{12}$$

and the answer is given by the coefficient of the term $r^2\,g\,b^2\,y$, which is

$$\frac{\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1} + 0 + 3 \times 2 \times 2 + 0}{12} = 16. \tag{13}$$

1. $(8 \times 4^2\,3^2\,2^2 + 3 \times 4^2\,3^2\,2^4 + 4^4\,3^4\,2^6)\,/\,12$.

Exercise 3. How many ways are there to color the six faces of a cube with six different colors? (Ans:[2])

Exercise 4. How many ways are there to color the six vertices of a regular hexagon with four colors, such that the number of vertices colored red is odd and at most two vertices are colored yellow?

Exercise 5. The nine squares of a 3×3 chessboard are painted red, white, and black. Assuming that the board can be rotated but not flipped.

a) Find the number of chessboards that have two white, two red, and five black squares;

b) Find the number of chessboards that have three squares of each color.

Exercise 6. In how many different ways can the faces of a cube be painted using three different colors if each color must be used at least once?

Exercise 7. Suppose that each vertex of a regular n-gon is labeled with either a 0 or a 1. For which values of n is the number of different labelings under the action of C_n equal to the number of different labelings under the action of D_n?

Exercise 8. Show that there are 114 ways of labeling the vertices of the cube with letters a,b,c,d if each letter occurs exactly twice.

2. 30.