1 The Permutation Group

We see that in the application of Burnside's Lemma, the key step is to determine how many elements of X are fixed under the action of a transformation g. So far we have to rely on our spatial imagination to do this step. It turns out that there is a more systematic way, discovered by Polya.

1.1 The permutation group

1.1.1 Definition and examples

Consider the problem of coloring the n faces of a certain polyhedron with m colors. Let g be a transformation that leaves the polyhedron unmoved. We would like to determine how many ways of coloring are there that stay unchanged under the action of g. The key observation here is that g is not just any transformation, it must move each face to some other face, and therefore is a member of the so-called *permutation group*.

Definition 1. (Permutation group) *A permutation group is a group consists of permutations of the elements of a certain set, with composition as the group operation.*

Example 2. Consider all permutations of $\{1, 2, 3\}$: $\pi_{1,2,3}$: $1 \to 1, 2 \to 2, 3 \to 3$; $\pi_{1,23}$: $1 \to 1, 2 \to 3, 3 \to 2$; π_{123}: $1 \to 2, 2 \to 3, 3 \to 1$; $\pi_{12,3}$: $1 \to 2, 2 \to 1, 3 \to 3$; $\pi_{13,2}$: $1 \to 3, 2 \to 2, 3 \to 1$; π_{132}: $1 \to 3, 2 \to 1, 3 \to 2$, together with composition as the binary operation, that is, for example, to determine $\pi_{13,2}\,\pi_{123}$, we check:

$$\pi_{123}: 1 \to 2, 2 \to 3, 3 \to 1; \qquad \pi_{13,2}: 1 \to 3, 2 \to 2, 3 \to 1. \tag{1}$$

Thus we have

$$(\pi_{13,2}\,\pi_{123})(1) = \pi_{13,2}(\pi_{123}(1)) = \pi_{13,2}(2) = 2. \tag{2}$$

$$(\pi_{13,2}\,\pi_{123})(2) = \pi_{13,2}(\pi_{123}(2)) = 1. \tag{3}$$

$$(\pi_{13,2}\,\pi_{123})(3) = \pi_{13,2}(\pi_{123}(3)) = 3. \tag{4}$$

Consequently we have

$$\pi_{13,2}\,\pi_{123}: 1 \to 2, 2 \to 1, 3 \to 3 \tag{5}$$

and therefore

$$\pi_{13,2}\,\pi_{123} = \pi_{12,3}. \tag{6}$$

We easily check that the six permutations now form a group. For example, $\pi_{1,2,3}$ is the identity element. $(\pi_{1,23})^{-1} = \pi_{1,23}$. We denote this group by S_3.

Exercise 1. Prove that S_3 is a group.

Notation 3. *We denote by S_n the group of all permutations of $\{1, 2, ..., n\}$, also called the symmetric group of $\{1, 2, ..., n\}$.*

Definition 4. (Subgroup) *Let G be a group. A subgroup H of G is a group with the following properties:*

i. All elements of H are elements of G;

ii. The binary operation of H is the binary operation of G.

Example 5. Recall the 5-balls-connected-by-4-rods problem. If we denote that balls from left to right (when assembled) by 1,2,3,4,5, then $H := \{i, f\}$ where

$$i: 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3, 4 \rightarrow 4, 5 \rightarrow 5, \qquad f: 1 \leftrightarrow 5, 2 \leftrightarrow 4, 3 \rightarrow 3 \tag{7}$$

is a subgroup of S_5.

Exercise 2. Prove this.

Example 6. The group of the 10 "cut-and-reconnect" operations for the Merry-Go-Rounds problem is a subgroup of S_{10}.

Exercise 3. Prove this.

Theorem 7. *Let H be a subgroup of G. Then the identity element i_H in H coincides with the identity element i_G in G.*

Exercise 4. Prove this theorem.

Theorem 8. *Let H be a subgroup of G. Then $|H|$ divides $|G|$.*

Proof. Let H act on G (as a set) through multiplication. Then G becomes a disjoint union of orbits. Let $g \in G$ be arbitrary. The orbit is $\text{Orb}_H(g) := \{h\, g | h \in H\}$. We claim that $|\text{Orb}_H(g)| = |H|$. It suffices to show that $h_1 \neq h_2 \Longrightarrow h_1\, g \neq h_2\, g$. But this is obvious as $h_i = (h_i\, g)\, g^{-1}$, $i = 1, 2$. □

Remark 9. Thus we see that any subgroup H of S_n satisfies that $|H|$ divides $n!$. Of course this would be true for any finite group H when n is large enough. So we may ask, whether every finite group is a subgroup of S_n for some n? Indeed this is so, thanks to Cayley's Theorem https://en.wikipedia.org/wiki/Cayley's_theorem.

1.1.2 Cauchy's two-line notation

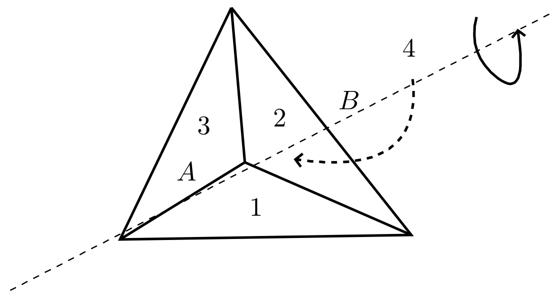
Let π be a permutation of $\{1, 2, ..., n\}$. An efficient notation for π is Cauchy's two-line notation:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \pi(1) & \pi(2) & \cdots & \pi(n-1) & \pi(n) \end{pmatrix}. \tag{8}$$

Example 10. Consider the 10-cart train. Let π be the operation of cutting between the 3rd and the 4th train and then connect the front segment to the back. We can write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}. \tag{9}$$

Example 11. Consider the regular tetrahedron and let π be the rotation of 180 degrees around the dotted line.



Then we can write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}. \tag{10}$$

Exercise 5. Find all operations that leaves a regular pentagon unmoved, and write every operation in Cauchy's two-line notation.

Exercise 6. Find all operations that leave a cube unmoved, and write every operation in Cauchy's two-line notation, treating the cube as

a) 6 faces marked 1,2,...,6; or

b) 8 vertices marked 1,2,...,8; or

c) 12 edges marked 1,2,...,12.

1.1.3 Cycles and cyclic form

There is a special kind of permutation, called cycles, that is of particular importance to our counting theory.

Definition 12. *Let π be a permutation of $\{1, 2, ..., n\}$. π is called a cycle if there are distinct elements $a_1, ..., a_k \in \{1, 2, ..., n\}$ such that*

$$\pi(a_1) = a_2, \quad \pi(a_2) = a_3, \quad \cdots \quad \pi(a_{k-1}) = a_k, \quad \pi(a_k) = a_1 \tag{11}$$

and $\pi(a) = a$ for all other $a \in \{1, 2, ..., n\}$. k is called the length of the cycle π.

Remark 13. For convenience, we allow $k = 1$. Of course a cycle of length 1 would simply be the identity.

Notation 14. *We denote a cycle simply as*

$$\pi = (a_1 a_2 \cdots a_k). \tag{12}$$

Exercise 7. Prove that $(a_1 a_2 \cdots a_k) = (a_2 a_3 \cdots a_k a_1)$.

The following theorem is intuitive. We leave the proof for interested readers.

Theorem 15. *Any permutation of $\{1, 2, ..., n\}$ is a product of disjoint cycles.*

Example 16. Consider $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$. It is a product of two cycles:

$$\pi = (\ 1\ \ 3\ )(\ 2\ \ 4\ ) = (\ 2\ \ 4\ )(\ 1\ \ 3\ ). \tag{13}$$

Remark 17. We see that, although the binary operation (composition) of permutations are in general not commutative, that is in general $\pi_1 \pi_2 \neq \pi_2 \pi_1$, the composition of disjoint cycles is commutative.

Exercise 8. Write the permutations in Exercise 5 as product of disjoint cycles.

1.2 Polya's theory

Consider coloring a device of n balls connected into some geometric shape through rods with m colors. Recall the general procedure:

1. Get $(m^n) \cdot n$ identical balls. Every time take n of them and mark $1, 2, ..., n$. Color each such group of n balls differently and assemble them into the desired geometric shape. We obtain m^n devices that are colored and marked. We call this collection X.

2. Each allowed operation turns one device into another if we ignore the colors but keep the marks. These operations form a group G.

3. The number of different devices we would have after erasing the marks is given through Burnside's Lemma:

$$\text{Ans} = \frac{1}{|G|} \sum_{g \in G} |X_g| \tag{14}$$

   where X_g is the collections of all marked devices that stays the same under the operation g, when we ignore the color.

Both Step 2 (determine the symmetry group for a certain geometric shape) and Step 3 could be non-trivial. There is very little we can do to Step 2, but Polya has found a formula for $|X_g|$, thus greatly simplifying Step 3.

Consider an arbitrary $g \in G$. As it turns one marked device into another, and a device is determined by the positions of the n marked balls, g is equivalent to a permutation π_g of $\{1, 2, ..., n\}$. Now we write π_g as cycles:

$$\pi_g = (a_1 a_2 \cdots a_k) \cdots. \tag{15}$$

Now let x be a coloring of of balls $1, 2, ..., n$ that does not change under the action of g. We clearly see that the balls $a_1, ..., a_k$ must be colored the same. In other words, there are exactly m different ways to color the balls $a_1, ..., a_k$. Application of the same logic to other cycle factors of π_g we reach the following.

Theorem 18. (Polya) *Let $g \in G$ be equivalent to a permutation π_g which is a product of l cycles, including cycles of length 1, then*

$$|X_g| = m^l. \tag{16}$$

We will see how this is applied in the next section.