Counting using Burnside's Lemma

Example 1. Let's recall the problem of coloring the 10-cart Merry-Go-Rounds with 2 colors.

Here X is the set of 2^{10} differently colored 10-cart trains. And $G = \{g_0, ..., g_9\}$ where g_k is the action: Cut the 10-cart train between the kth and the $(k+1)$th carts, and then re-connect the front half to the end of the back half.

By Burnside's Lemma, we have

$$\text{Ans} = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{10} \sum_{i=0}^{9} |X_{g_i}|. \tag{1}$$

Now we calculate

- X_0. Since g_0 is the identity element, we have $X_0 = X$ and therefore $|X_0| = 1024$.
- X_1. A 10-cart train remains the same after the action of g_1 if and only if it is colored by a single color and therefore $|X_1| = 2$.
- X_2. A 10-cart train remains the same after the action of g_2 if and only if carts 1,3,5,7,9 are colored the same and 2,4,6,8,10 are colored the same. So $|X_2| = 4$.
- X_3. A 10-cart train remains the same after the action of g_3 if and only if all carts are colored the same. So $|X_3| = 2$.
- $|X_4| = 4$.
- $|X_5| = 32$.
- $|X_6| = 4$.
- $|X_7| = 2$.
- $|X_8| = 4$.
- $|X_9| = 2$.

Therefore the answer is

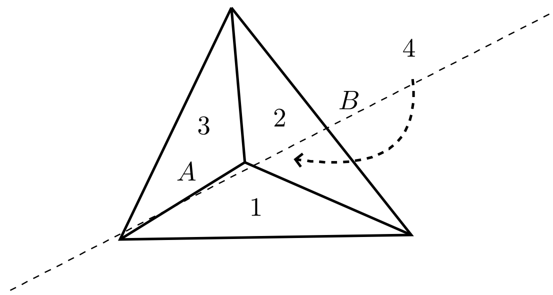
$$\frac{1024 + 2 + 4 + 2 + 4 + 32 + 4 + 2 + 4 + 2}{10} = 108. \tag{2}$$

Exercise 1. How many ways are there to color a 6-cart Merry-Go-Round with 4 colors?

Example 2. Let $m \in \mathbb{N}$. How many ways are there to color the 4 faces of a regular tetrahedron with m colors?

Solution. If we mark the faces, then we can make m^4 differently colored tetrahedra. These would form our set X. On the other hand, the group G that acts on X would be the symmetry group of a regular tetrahedron, that is, spatial rigid movements (translation, rotation, reflection and their combinations) that leave the tetrahedron unmoved.

Now we determine the symmetry group of a regular tetrahedron. Let's put the tetrahedron on the x-y plane. Mark the faces 1,2,3,4 and let face 4 be the down face, and let 1,2,3 be counterclockwise when looking from above.



Then there are three transformations that leaves the tetrahedron unmoved and with face 4 still the “down” face: counterclockwise rotation around the z-axis by $0, 2\pi/3, 4\pi/3$. We denote them $r_{40}, r_{4,123}, r_{4,132}$. Note that $r_0 = i$ the identity. Also the “123” in the subscript means face 1 is rotated to face 2, face 2 to face 3 (and of course face 3 to face 1). Now as the four faces are identical, we see that there are three transformations that leaves both the tetrahedron and face 3 unmoved, denote them $r_{30} = i$, $r_{3,124}$, $r_{3,142}$. Similarly we have transformations that leave faces 1 or 2 unmoved. So far we have 9 transformations:

$$i, r_{4,123}, r_{4,132}, r_{3,124}, r_{3,142}, r_{2,134}, r_{2,143}, r_{1,243}, r_{1,234}. \tag{3}$$

Now consider transformations that move all four faces. It is clear that we can rotate around the line passing A, B to achieve $1 \leftrightarrow 3$, $2 \leftrightarrow 4$. We denote it by $r_{13,24}$. Similarly there are $r_{14,23}$, $r_{12,34}$. We now have 12 transformations.

Exercise 2. What is $r_{4,123} r_{12,34}$?

Are there more? Assume there is. Without loss of generality, assume face 1 becomes face 2. If $2 \to 1$ then necessarily $3 \leftrightarrow 4$ and we end up with $r_{12,34}$. Thus either $2 \to 3$ or $2 \to 4$. The two cases are clearly equivalent due to the symmetry of the regular tetrahedron. Without loss of generality assume $2 \to 3$. If $3 \to 1$ then we have $r_{4,123}$. Thus the only possibility for a new transformation is $1 \to 2 \to 3 \to 4 \to 1$. we have Thus we have $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ for the faces. We now show that this transformation is impossible. To see this, note that when we are at the vertex facing 4, the faces 1,2,3 and counter-clockwise. This orientation should not change if we stay at this vertex and move with the tetrahedron during the transformation. However, after the transformation $1 \to 2 \to 3 \to 4 \to 1$, we are at the vertex facing 1, but the faces 2,3,4 are not clockwise. Contradiction.

Therefore our symmetry group G has 12 elements: $i, r_{4,123}, r_{4,132}, r_{3,124}, r_{3,142}, r_{2,134}, r_{2,143}, r_{1,243}, r_{1,234}, r_{12,34}, r_{13,24}, r_{14,23}$.

We now calculate X_g for every one of these element.

- i. $|X_g| = |X| = m^4$.
- $r_{4,123}$. Necessarily faces 1,2,3 have the same color. There are two cases.
  - All 4 faces are colored the same. There are m such colorings.
  - Two colors are used. There are $2 \times \binom{m}{2}$ such colorings.[1]

  So $|X_{r_{4,123}}| = m(m-1) + m = m^2$.
- $r_{4,132}, \ldots, r_{1,234}$. Same as the $r_{4,123}$ case.

1. After two colors are chosen, we can further choose which one to use for face 4.

- $r_{12,34}$. Faces 1,2 have the same color and faces 3,4 have the same color. There are two cases.
  - All 4 faces are colored the same. There are m such colorings.
  - Two colors are used. There are $2 \times \binom{m}{2}$ such colorings.

  So $|X_{r_{12,34}}| = m^2$.
- $r_{13,24}, r_{14,23}$. Same as $r_{12,34}$.

Thus we have the total number of colorings to be

$$C(m) := \frac{m^4 + 11\,m^2}{12}. \tag{4}$$

Exercise 3. Prove directly (like in number theory) that $12|\,(m^4 + 11m^2)$. (Hint: [2])

Remark 3. We see that

$$C(2) = 5 = \binom{4+2-1}{2-1}, \qquad C(3) = 15 = \binom{4+3-1}{3-1}, \qquad C(4) = 36 \neq 35 = \binom{4+4-1}{4-1}. \tag{5}$$

So it is a coincidence that in the midterm, the number of ways to color the 4 faces with 3 colors is the same as the number of ways to color 4 identical balls in 3 colors.

Example 4. How many ways are there to color a 4-bead bracelet with 3 colors if

a) It is not allowed to "flip" the bracelet.

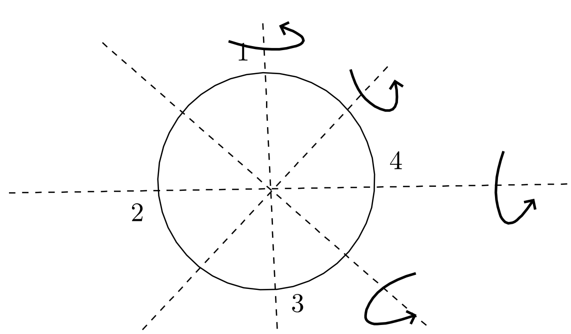
b) It is allowed to "flip" the bracelet.

Solution. The set X has 3^4 elements.

a) When it is not allowed to "flip", the only possible transformations are the four rotations by $0, \pi/2, \pi$, $3\,\pi/2$. We see that the answer is

$$\frac{3^4 + 3^1 + 3^2 + 3^1}{4} = 24. \tag{6}$$

b) When "flipping" is allowed, the situation is more complicated. Besides the four rotations we have also four "flippings":



2. Write as $(m^2 + 11)\,m^2$ and show that it is divisible by 3 through discussing $m = 3\,k, 3\,k+1, 3\,k+2$. Similarly show that it is divisible by 4 through discussing $m = 4\,k, 4\,k+1, 4\,k+2, 4\,k+3$.

We discuss the following cases.

- $1 \to 1$. As the distance (along the circle) between the beads must be unchanged, either $2 \to 2$, $4 \to 4$ which leads to the identity transformation, or $2 \leftrightarrow 4$ which is “flipping” around the 1-3 axis.

- $1 \to 3$. Either $4 \leftrightarrow 2$ which is rotation by π, or $4, 2$ stay unmoved which is flipping around the 2-4 axis.

- $1 \to 2$. Either $4 \to 1$ which means $2 \to 3$ and $3 \to 4$, which is rotation by $\pi/2$; or $4 \to 3$ which dictates $2 \to 1$ and $3 \to 4$. This is flipping around the diagonal axis connecting the middle of 1,2 and 3,4.

Thus we see that these eight transformations form the symmetry group of the bracelet. It is easy to calculate the $|X_g|$'s and the answer is

$$\frac{3^4 + 3^3 + 3^3 + 3^2 + 3^2 + 3^2 + 3^1 + 3^1}{8} = 21. \tag{7}$$