Example 1. Let $a_{n+2} = 3\,a_{n+1} - 2\,a_n$ for $n \geqslant 0$, and let $a_0 = 0$, $a_1 = 1$. Find an explicit formula for a_n.

Solution. Note that the recurrence relation can be written as

$$a_n = 3\,a_{n-1} - 2\,a_{n-2}, \qquad n \geqslant 2. \tag{1}$$

Setting $A(x) = \sum_{n=0}^{\infty} a_n\,x^n$ we have

$$\begin{aligned} A(x) &= a_0 + a_1\,x + \sum_{n=2}^{\infty} a_n\,x^n \\ &= x + \sum_{n=2}^{\infty} (3\,a_{n-1} - 2\,a_{n-2})\,x^n \\ &= x + 3\,x \sum_{n=2}^{\infty} a_{n-1}\,x^{n-1} - 2\,x^2 \sum_{n=2}^{\infty} a_{n-2}\,x^{n-2} \\ &= x + 3\,x \sum_{n=1}^{\infty} a_n\,x^n - 2\,x^2 \sum_{n=0}^{\infty} a_n\,x^n \\ &= x + 3\,x \sum_{n=0}^{\infty} a_n\,x^n - 2\,x^2 \sum_{n=0}^{\infty} a_n\,x^n \\ &= x + 3\,x\,A(x) - 2\,x^2\,A(x). \end{aligned} \tag{2}$$

Thus

$$A(x) = \frac{x}{1 - 3\,x + 2\,x^2}, \tag{3}$$

from which we can apply partial fraction expansion to obtain $a_n = 2^n - 1$.

It is also possible to sove recurrence relations that are not homogeneous.

Example 2. Solve $h_n = 2\,h_{n-1} + 3^n$, $n \geqslant 1$ and $h_0 = 2$.

Solution. Let $H(x) = \sum_{n=0}^{\infty} h_n\,x^n$. We have

$$\begin{aligned} H(x) &= h_0 + \sum_{n=1}^{\infty} h_n\,x^n \\ &= 2 + \sum_{n=1}^{\infty} (2\,h_{n-1} + 3^n)\,x^n \\ &= 2 + 2 \sum_{n=1}^{\infty} h_{n-1}\,x^n + \sum_{n=1}^{\infty} 3^n\,x^n \\ &= 2 + 2 \sum_{n=0}^{\infty} h_n\,x^{n+1} + \frac{3\,x}{1 - 3\,x} \\ &= 2\,x\,H(x) + \frac{2 - 3\,x}{1 - 3\,x}. \end{aligned} \tag{4}$$

Therefore

$$H(x) = \frac{2 - 3\,x}{(1 - 2\,x)\,(1 - 3\,x)}. \tag{5}$$

Application of partial fraction expansion gives

$$H(x) = \frac{-1}{1 - 2\,x} + \frac{3}{1 - 3\,x}. \tag{6}$$

Consequently $h_n = -2^n + 3^{n+1}$.

Exercise 1. Solve

$$h_n = h_{n-1} + n^3, \qquad h_0 = 0. \tag{7}$$

(Ans: [1])

Exercise 2. Solve

$$h_n = 3\,h_{n-1} + 3^n, \qquad h_0 = 2. \tag{8}$$

(Ans:[2])

Before ending this unit, we check out some non-trivial examples.

Example 3. All n soldiers of a military squadron stand in a line. The officer in charge splits the line at several places, forming smaller (nonempty) units. Then he names one person in each unit to be the commander of that unit. Let h_n be the number of ways he can do this. Find a closed formula for h_n.

Solution. Say the n soldiers are split at $l-1$ places, forming segments of $k_1, \ldots, k_l$ soldiers. In this situation there are clearly $k_1 \times \cdots \times k_l$ different ways to choose the commanders. Therefore, if the soldiers are split into l segments, the number of ways to choose the commanders is given by the coefficient for x^n in the expansion of

$$\left(\sum_{k=0}^{\infty} k\,x^k \right)^l \tag{9}$$

Consequently, when we consider all possible l, we see that the generating function for h_n is

$$H(x) = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{\infty} k\,x^k \right)^l = \sum_{l=0}^{\infty} \left(\frac{x}{(1-x)^2} \right)^l = \frac{1}{1 - \frac{x}{(1-x)^2}} = 1 + \frac{x}{1 - 3\,x + x^2} \tag{10}$$

Application of the partial fraction expansion now gives

$$\frac{1}{1 - 3\,x + x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{x - \alpha} - \frac{1}{x - \beta} \right) \tag{11}$$

where $\alpha = \frac{(3 + \sqrt{5})}{2}$, $\beta = \frac{3 - \sqrt{5}}{2}$. Finally the answer is

$$h_n = \frac{1}{\sqrt{5}}\,(\alpha^n - \beta^n). \tag{12}$$

Example 4. Let h_n be the number of ways of dividing a regular $(n+1)$-gon into triangles by inserting diagonals that do not intersect in the interior.

1. $\frac{n^2\,(n+1)^2}{4}$.
2. $(2+n)\,3^n$.

Then we have (set $h_1 = 1$)

- The following recurrence relation holds for $n \geqslant 2$:

$$h_n = h_1 h_{n-1} + h_2 h_{n-2} + \cdots + h_{n-1} h_1. \tag{13}$$

  To see this, we pick an arbitrary side and fix it as the "base". There is a triangle containing this side seperating the $(n+1)$-gon into three parts: An $(l+1)$-gon to the left of this triangle, the triangle, and an $(n-l+1)$-gon to the right of this triangle.

- Let $h(x) = \sum_{n=1}^{\infty} h_n x^n$. Then (13) becomes

$$\begin{aligned} h(x) &= h_1 x + \sum_{n=2}^{\infty} h_n x^n \\ &= h_1 x + \sum_{n=2}^{\infty} (h_1 x \cdot h_{n-1} x^{n-1} + h_2 x^2 \cdot h_{n-2} x^{n-2} + \cdots + h_{n-1} x^{n-1} \cdot h_1 x) \\ &= x + h(x)^2. \end{aligned} \tag{14}$$

  The generating function thus satisfies

$$h(x)^2 - h(x) + x = 0. \tag{15}$$

- We see that

$$h(x) = \frac{1 \pm (1 - 4x)^{1/2}}{2}. \tag{16}$$

  As $h(0) = 0$, we have to pick the "$-$" sign and consequently

$$h(x) = \frac{1}{2} - \frac{1}{2} (1 - 4x)^{1/2}. \tag{17}$$

- Now we do Taylor expansion of $h(x)$. We have

$$h(x) = \frac{1}{2} - \frac{1}{2} (1 - 4x)^{1/2} \implies h(0) = 0; \tag{18}$$

$$h'(x) = (1 - 4x)^{-1/2} \implies h'(0) = 1; \tag{19}$$

$$h''(x) = 2 (1 - 4x)^{-3/2} \implies h''(0) = 2; \tag{20}$$

$$h'''(x) = 12 (1 - 4x)^{-5/2} \implies h'''(0) = 12; \tag{21}$$

  It is now clear that

$$h^{(n)}(x) = c_n (1 - 4x)^{\frac{1}{2} - n}. \tag{22}$$

  To figure out c_n, we differentiate again to have

$$h^{(n+1)}(x) = 2 (2n - 1) c_n (1 - 4x)^{\frac{1}{2} - n - 1}. \tag{23}$$

  Therefore

$$c_{n+1} = 2 (2n - 1) c_n = 2^2 (2n - 1) (2n - 3) c_{n-1} = \cdots = 2^n (2n - 1) \cdots (1), \tag{24}$$

  that is

$$c_n = 2^{n-1} (2n - 3) \cdots 1. \tag{25}$$

- This gives

$$h_n = \frac{c_n}{n!} = \frac{1}{n} \frac{2^{n-1} (2n-3) \cdots 1}{(n-1) \cdots 1} = \frac{1}{n} \binom{2n-2}{n-1}, \qquad n \geqslant 2. \tag{26}$$

  Called “Catalan number”.

We end this unit with some curious facts about the Stirling numbers (of the 2nd kind) and the Bell numbers.

Example 5. We recall that the Stirling numbers of the 2nd kind has recurrence relation

$$S(n,m) = S(n-1,m-1) + m\, S(n-1,m). \tag{27}$$

We form the generating function

$$S_m(x) := \sum_{n=0}^{\infty} S(n,m)\, x^n. \tag{28}$$

Thus we have

$$\begin{aligned} S_m(x) &= \sum_{n=1}^{\infty} S(n,m)\, x^n \\ &= \sum_{n=1}^{\infty} S(n-1,m-1)\, x^n + \sum_{n=1}^{\infty} m\, S(n-1,m)\, x^n \\ &= x\, S_{m-1}(x) + m\, x\, S_m(x). \end{aligned} \tag{29}$$

Therefore

$$S_m(x) = \frac{x}{1-m\,x} S_{m-1}(x) = \cdots = \frac{x^m}{(1-x)(1-2x)\cdots(1-m\,x)}. \tag{30}$$

We apply partial fraction:

$$\frac{1}{(1-x)(1-2x)\cdots(1-m\,x)} = \frac{A_1}{1-x} + \frac{A_2}{1-2x} + \cdots + \frac{A_m}{1-m\,x}. \tag{31}$$

It is easy to see that

$$A_l = \frac{1}{\left(1-\frac{1}{l}\right)\left(1-\frac{2}{l}\right)\cdots\left(1-\frac{l-1}{l}\right)\left(1-\frac{l+1}{l}\right)\cdots\left(1-\frac{m}{l}\right)} = (-1)^{m-l} \frac{l^{m-1}}{(l-1)!\,(m-l)!}. \tag{32}$$

Therefore

$$S(n,m) = \sum_{l=1}^{m} (-1)^{m-l} \frac{l^n}{l!\,(m-l)!}. \tag{33}$$

Of course this is simply $T(n,m)/m!$.

Exercise 3. Prove that

$$\sum_{l=1}^{m} (-1)^{m-l} \frac{l^n}{l!\,(m-l)!} = 0 \tag{34}$$

if $m > n$. (Hint:[3])

Example 6. Recall that the Bell numbers $B_n = S(n,1) + \cdots + S(n,n)$. We have

$$\begin{aligned} B_n &= \sum_{m=1}^{n} \sum_{l=1}^{m} (-1)^{m-l} \frac{l^{n-1}}{(l-1)!\,(m-l)!} \\ &= \sum_{m=1}^{M} \sum_{l=1}^{m} (-1)^{m-l} \frac{l^{n-1}}{(l-1)!\,(m-l)!} \\ &= \sum_{l=1}^{M} \sum_{m=l}^{M} (-1)^{m-l} \frac{l^{n-1}}{(l-1)!\,(m-l)!} \\ &= \sum_{l=1}^{M} \frac{l^{n-1}}{(l-1)!} \left\{ \sum_{s=0}^{M-l} \frac{(-1)^s}{s!} \right\}. \end{aligned} \tag{35}$$

Taking $M \longrightarrow \infty$ we have

$$B_n = \frac{1}{e} \sum_{l \geqslant 0} \frac{l^n}{l!}. \tag{36}$$

Now we try to find the generating function for B_n. Let

$$B(x) := \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n. \tag{37}$$

Thus we have

$$\begin{aligned} B(x) - 1 &= \frac{1}{e} \sum_{n \geqslant 1} \frac{x^n}{n!} \sum_{l \geqslant 1} \frac{l^{n-1}}{(l-1)!} \\ &= \frac{1}{e} \sum_{l \geqslant 1} \frac{1}{l!} (e^{rx} - 1) \\ &= e^{e^x - 1} - 1. \end{aligned} \tag{38}$$

Thus the exponential generating function for the Bell numbers is

$$B(x) = e^{e^x - 1}. \tag{39}$$

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3. Note that (34) follows from (30) where it is clear that the power of x^n is zero for all $n < m$.