Exponential generating functions

Definition 1. *The **exponential generating function** for a sequence of numbers $a_0, a_1, \ldots$ is*

$$E(x) := a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \cdots + \frac{a_n x^n}{n!} + \cdots \tag{1}$$

Example 2. Let m be fixed. Find a generating function of the sequence $a_n := T(n, m)$.

Solution. Recall that $T(n, m)$ is the number of ways distributing n different balls into m different boxes, leaving no box empty. This can be done as follows: Let n_1, ..., $n_m > 0$ be such that $n_1 + \cdots + n_m$. We put n_1 balls into box 1, n_2 balls into box 2, ..., n_m balls into box m. Thus when $n_1, \ldots, n_m$ are fixed, there are

$$\binom{n}{n_1, \ldots, n_m} = \frac{n!}{n_1! \cdots n_m!} \tag{2}$$

different ways of doing this. The total number of different ways is then given by

$$\sum_{n_1, \ldots, n_m > 0, n_1 + \cdots + n_m = n} \frac{n!}{n_1! \cdots n_m!}. \tag{3}$$

This inspires us to use exponential generating functions. We notice that the coefficient for x^n in the expansion of

$$\left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^m \tag{4}$$

is exactly

$$\sum_{n_1, \ldots, n_m > 0, n_1 + \cdots + n_m = n} \frac{1}{n_1! \cdots n_m!}. \tag{5}$$

Therefore we see that the exponential generating function for $T(n, m)$ is

$$\begin{aligned} \sum_{n=m}^{\infty} \frac{T(n, m)}{n!} x^n &= \left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^m \\ &= (e^x - 1)^m \\ &= e^{mx} - \binom{m}{1} e^{(m-1)x} + \binom{m}{2} e^{(m-2)x} - \cdots + (-1)^{m-1} \binom{m}{m-1} e^x + (-1)^m. \end{aligned} \tag{6}$$

From which it follows that

$$T(n, m) = m^n - \binom{m}{1} (m-1)^n + \cdots + (-1)^{m-1} \binom{m}{m-1} \tag{7}$$

Remark 3. Exponential generating functions are used for problems equivalent to distributing different balls into boxes. Another way to understand this is that when the sequence is related to permutations, it may be a good idea to use exponential generating functions.

Example 4. Compute the number of sequences of length 10 that can be formed using 5 different symbols.

Solution. Denote the 5 symbols A,B,C,D,E. Then the problem is equivalent to putting 10 different balls into 5 different boxes. The exponential generating function is

$$\left(1+x+\frac{x^2}{2!}+\cdots\right)^5=e^{5x} \tag{8}$$

so the answer is 5^{10}.

Example 5. Find the number of sequences of length 8 that can be formed using 1,2, or 3 a's; 2,3, or 4 b's; and 0,2, or 4 c's.

Solution. This is equivalent to distributing 8 different balls into 3 boxes marked A, B, C, with the requirement that there are 1, 2, or 3 balls in box A, 2, 3, or 4 balls in box B, and 0, 2, or 4 balls in box C. Therefore we use the exponential generating function

$$E(x)=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}\right). \tag{9}$$

Inspection of the formula shows that the x^8 term is given by

$$x\cdot\frac{x^3}{3!}\cdot\frac{x^4}{4!}+\frac{x^2}{2!}\left[\frac{x^2}{2!}\cdot\frac{x^4}{4!}+\frac{x^4}{4!}\cdot\frac{x^2}{2!}\right]+\frac{x^3}{3!}\cdot\frac{x^3}{3!}\cdot\frac{x^2}{2!} \tag{10}$$

which means the answer is

$$\frac{8!}{3!\,4!}+\frac{8!}{2!\,4!}+\frac{8!}{3!\,3!\,2!}=280+840+560=1680. \tag{11}$$

Example 6. Find the number of ways to distribute n different objects to five different boxes if

a) An even number of objects are distributed to box 5.

b) A positive even number of objects are distributed to box 5.

Solution.

a) The generating function is

$$\left(1+x+\frac{x^2}{2!}+\cdots\right)^4\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)=(e^x)^4\left(\frac{e^x+e^{-x}}{2}\right) \tag{12}$$

so the answer is $(5^n+3^n)/2$.

b) The generating function is

$$(e^x)^4\left[\left(\frac{e^x+e^{-x}}{2}\right)-1\right] \tag{13}$$

so the answer is $(5^n+3^n)/2-4^n$.

Example 7. Use exponential generating function to find the number of n-digit sequences that can be constructed from the digits $\{0,1,2,3\}$ for which the total number of 0's and 1's is even.

Solution. If we ignore the last requirement, the generating function is simply

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^4=\sum_{n=0}^{\infty}\left(\sum_{n_i\geqslant 0,n_0+\cdots n_3=n}\frac{1}{n_0!\cdots n_3!}\right)x^n. \tag{14}$$

The last requirement now translates to $n_0 + n_1 = \text{even}$. Noticing that

$$\begin{aligned}\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^4 &= \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2 \\ &= \left(\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\frac{1}{n_0!\,n_1!}x^{n_0+n_1}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2, \qquad (15)\end{aligned}$$

We see that the exponential generating function for the problem is given by

$$\left(\text{Even powers in the expansion of }\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2\right)\cdot\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2 \qquad (16)$$

which, thanks to $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2 = e^{2x}$, is

$$\frac{e^{2x}+e^{-2x}}{2}\,e^{2x} = \frac{e^{4x}+1}{2}. \qquad (17)$$

Therefore the answer is 2^{2n-1}.

Exercise 1. Find the number of n-digit sequences that can be constructed from the digits $\{0,1,2,3\}$ for which the total number of 0's and 1's is a multiple of 4. You need to know complex numbers to solve this problem. (Hint:[1])

Exercise 2. Use generating function to calculate the number of ways to arrange 6 letters chosen from the word MISSISSIPPI.

Exercise 3. Use an exponential generating function to find the number of n-digit ternary sequences in which no digit appears exactly once.

Exercise 4. Use exponential generating function to find the number of ways to distribute 10 different toys. to 4 different children if

a) The first child receives at least one toy;

b) The second child receives at least two toys;

c) The first child receives at least one toy and the second child receives at least two toys.

Coloring problems

Example 8. Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors red, white, and blue, if an even number of squares are to be colored red.

Solution. We have the exponential generating function to be

$$\begin{aligned}E(x) &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^2 \\ &= \frac{e^x+e^{-x}}{2}\,e^{2x} \\ &= \frac{1}{2}\,(e^{3x}+e^x) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty}3^n\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(3^n+1)\frac{x^n}{n!}. \qquad (18)\end{aligned}$$

1. $e^x + e^{ix} + e^{-x} + e^{-ix} = ?$.

Therefore the answer is $(3^n+1)/2$.

Exercise 5. Determine the number h_n of n-digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times. (Ans:[2])

Exercise 6. Determine the number h_n of ways to color the squares of a 1-by-n board with the colors red, white, and blue, where the number of red squares is even and there is at least one blue square. (Ans:[3])

2. $(5^n+2\times 3^n+1)/4$.
3. $(3^n-2^n+1)/2$.