Motivating examples

Example 1. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. We notice that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1}$$

Setting $x = 1$ gives the result. □

Example 2. Prove that the number of different partitions with odd summands of n is the same as the number of different partitions with distinct summands.

Proof. If we let a_n denote the first number and b_n denote the second number, then we have

$$\sum_{n=0}^{\infty} a_n x^n = (1 + x + x^2 + \cdots)(1 + x^3 + \cdots) \cdots = \frac{1}{1-x} \frac{1}{1-x^3} \cdots \tag{2}$$

while

$$\sum_{n=0}^{\infty} b_n x^n = (1+x)(1+x^2)(1+x^3)\cdots. \tag{3}$$

Then all we need to show is

$$\frac{1}{1-x} \frac{1}{1-x^3} \cdots = (1+x)(1+x^2)(1+x^3)\cdots. \tag{4}$$

This is done through the following trick.

$$\begin{aligned}(1+x)(1+x^2)(1+x^3) &= \frac{1-x^2}{1-x} \frac{1-x^4}{1-x^2} \frac{1-x^6}{1-x^3} \cdots \\ &= \frac{1}{1-x} \frac{1}{1-x^3} \cdots \end{aligned} \tag{5}$$

Note that the terms with even powers of x cancel each other. □

Example 3. There are 3 identical weights of 1g each, 4 identical weights of 2g each, and 2 identical weights of 4g each. How many different ways are there to obtain 6g from these weights?

Solution. The answer is given by the coefficent of x^6 in the expansion of

$$(1 + x + x^2 + x^3)(1 + x^2 + x^4 + x^6 + x^8)(1 + x^4 + x^8). \tag{6}$$

We re-write as

$$(1-x^4)(1-x^{10})(1-x^{12}) \left[\frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^4} \right] \tag{7}$$

and then apply partial fraction to $[\cdots]$. We will review how to do this in the next lecture.

Exercise 1. What if the weights are all different? (Ans.[1])

1. $(1+x)^3 (1+x^2)^4 (1+x^4)^2$.