1. Review of basics

1.1. Basic counting

- Product and sum rules.
  - Product rule.
    1. Ordered steps (distinguishable steps);
    2. Independent choices;
    3. Distinguishable outcomes.
  - Sum rule.
    1. Disjoint;
    2. Union is the whole.

Example 1. Consider the product

$$(a+b+c)\,(d+e+f)\,(p+q+r+s)\,(x+y+u+v+w). \tag{1}$$

a) Do the following terms appear in the expansion:

$$adps, \qquad blsw, \qquad bfpu, \qquad bfxw. \tag{2}$$

b) How many terms are there in the expansion?

Solution.

a) No. No. Yes. No.

b) $3\times 3\times 4\times 5=180$.

- Inclusion-exclusion.

  Let A_1 be objects satisfying property 1, A_2 be objects satisfying property 2, ..., A_n be objects satisfying property n. Then $A_1\cup\cdots\cup A_n$ is the collection of objects satisfying at least one of the properties $1,2,\ldots,n$. This is hard to count. On the other hand, it is often easy to count intersections of the A_i's, such as $A_1\cap A_2$ which is the collection of objects satisfying both properties 1 and 2. This is the motivation for the following inclusion-exclusion principle.

$$\begin{aligned}|A_1\cup\cdots\cup A_n| &= \sum_{i=1}^n |A_i| \\ &\quad -\sum_{i,j=1,i\neq j}^n |A_i\cap A_j| \\ &\quad +\sum_{i,j,k=1,i,j,k\text{ distinct}}^n |A_i\cap A_j\cap A_k| \\ &\quad -\cdots+(-1)^{n-1}|A_1\cap\cdots\cap A_n|.\end{aligned} \tag{3}$$

Example 2. How many integers from $1,\ldots,1000$ is divisible by 9, 11, or 13?

Solution. We set

$$\begin{aligned}A_1 &:= \text{integers from } 1,\ldots,1000 \text{ is divisible by } 9.\\ A_2 &:= \text{integers from } 1,\ldots,1000 \text{ is divisible by } 11.\\ A_3 &:= \text{integers from } 1,\ldots,1000 \text{ is divisible by } 13\end{aligned}$$

Thus we have

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3| &= \left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor + \left\lfloor \frac{1000}{13} \right\rfloor \\
&\quad - \left\lfloor \frac{1000}{99} \right\rfloor - \left\lfloor \frac{1000}{117} \right\rfloor - \left\lfloor \frac{1000}{143} \right\rfloor + \left\lfloor \frac{1000}{1287} \right\rfloor \\
&= 252.
\end{aligned} \tag{4}
$$

Example 3. A single die is rolled five times in a row. How many outcomes will have the fifth number equal to an earlier number?

Solution. We set

$$
\begin{aligned}
A_1 &:= \text{The 5th is the same as the 1st;} \\
A_2 &:= \text{The 5th is the same as the 2nd;} \\
A_3 &:= \text{The 5th is the same as the 3rd;} \\
A_4 &:= \text{The 5th is the same as the 4th.}
\end{aligned}
$$

We see that $|A_1| = |A_2| = |A_3| = |A_4| = 6^4$. Furthermore we see that $|A_i \cap A_j| = 6^3$ whenever $i \neq j$, $|A_i \cap A_j \cap A_k| = 6^2$, and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 6$. Next notice that there are $C(4,2)$ different $A_i \cap A_j$ and $C(4,3)$ different $A_i \cap A_j \cap A_k$. Therefore

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| &= 4 \times 6^4 - 6 \times 6^3 + 4 \times 6^2 - 6 \\
&= 4026.
\end{aligned} \tag{5}
$$

1.2. Permutations and combinations

- How many different ways are there to list n distinct objects in a line?

$$P(n) := n! = n \cdot (n-1) \cdot \dots \cdot 1. \tag{6}$$

- How many different ways are there to list m objects from n distinct objects into a line?

$$P(n,m) := \frac{n!}{(n-m)!}. \tag{7}$$

- How many different ways are there to list n objects which consists of k groups of $n_1, \ldots, n_k$ identical objects respectively?

$$\binom{n}{n_1, \ldots, n_k} := \frac{n!}{n_1! \cdots n_k!}. \tag{8}$$

  - $k = 2$, $n_1 = m$, $n_2 = n - m$. Combination numbers

$$C(n,m) = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}. \tag{9}$$

- Alternative interpretations.
  - $C(n,m)$: How many different ways to pick m objects from n different objects?
  - $\binom{n}{n_1, \ldots, n_k}$: How many different ways to divide n different objects into k groups of n_1, ..., n_k objects each?

Example 4. Signals are made by running five colored flags up a mast.

a) How many different signals can be made if there is an unlimited supply of flags of seven different colors?

b) What if adjacent flags in a signal must not be of the same color?

c) What if all five flags in a signal must be of different colors?

Solution.

a) We do this in five steps, in the ith step the ith flag chooses its color. By product rule there are 7^5 signals.

b) The 1st flag can choose from 7 colors. The following flags can choose from 6 colors. Thus the answer is 7×6^4.

c) In this case we are ordering 5 flags from 7 different flags, the answer is then $P(7,5) = 2520$.

Example 5. There are nine different books on a shelf; four are red and five are green. In how many different orders is it possible to arrange the books on the shelf if

a) there are no restrictions;

b) the red books must be together and the green books together;

c) the green books must be together whereas the red books may be, but need not be, together;

d) the colors must alternate, i.e. no two books of the same color may be adjacent?

e) What if except for difference in color, the books are identical?

f) What if the red books are identical but the green books are different?

Solution.

a) This is the same as lining up 9 different objects. $P(9) = 9!$.

b) There are two cases. red books - green books, and then green books-red books. In each case we carry out two steps.

   1. Order the red books. There are 4! different ways.
   2. Order the green books. There are 5! different ways.

   By the product rule and the sum rule, the answer is $4! \times 5! + 4! \times 5! = 5760$.

c) We first order the green books, in any of the 5! possible ways, then bind the green books together and order them together with the 4 red books. By the product rule the answer is $5! \times 5! = 14400$.

d) We first order the 5 green books, which has 5! different ways. Then we put the 4 red books into the 4 "in-between spaces". There are 4! different ways. The total, by the product rule, is $5! \times 4! = 2880$.

e) This is the same as picking 4 out of 9 different positions for the red books, and the rest for the green books. $C(9,4) = 126$.

f) We do this in two steps.

   1. Ignore the difference between green books. There are $C(9,4)$ different ways to do this.
   2. Re-order the green books. There are 5! different ways to do this.

   Thus by the product rule the answer is $5! \times C(9,4) = 15120$.