The boxes are allowed to be empty

- The last case in the occupancy problem is putting n identical balls into m identical boxes and allow boxes to be empty.
- The answer is clearly given by

$$P_m(n) := p_1(n) + p_2(n) + \cdots + p_m(n). \tag{1}$$

- We claim that $P_m(n) = p_m(n+m)$. To see this, we distribute $n+m$ identical balls into m boxes, nonempty, in two steps.

  1. In the first step we take m balls and put them into m boxes, one ball in each box. As the balls are identical and the boxes are identical, there is exactly one way to do this.
  2. In the second step we take the remaining n balls and put them into m boxes without any requirement. There are $P_m(n)$ different ways to do this.

  By the product rule we see that

$$p_m(n+m) = 1 \times P_m(n) = P_m(n). \tag{2}$$

Simple Coloring Problems

Consider the following problem.

> We have n beads and m colors. How many ways are there to color the beads?

Infinite supply of colors

Problems the does not involve symmetry

Example 1. How many ways are there to color 5 different beads with infinite supply of 2 colors?

Solution. We see that this is the same as putting n different beads into m different boxes, and then color the beads in the first box by color one, the beads in the second box by color two, and so on. Thus the answer is $2^5 = 32$.

Alternatively, we can color the beads in 5 steps. In step i, the ith bead chooses a color for itself.

Exercise 1. How many ways are there to color n different beads with infinite supply of m colors?

Exercise 2. Check that if we denote the number of ways to color n different beads with infinite supply of m colors by A_n, then

$$A_0 + A_1 x + \frac{A_2}{2!} x^2 + \cdots + \frac{A_n}{n!} x^n + \cdots = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^m. \tag{3}$$

Example 2. How many ways are there to color n identical beads with infinite supply of m colors?

Solution. We see that this is the same as putting n identical balls into m different boxes. Thus the answer is $\binom{n+m-1}{m-1}$.

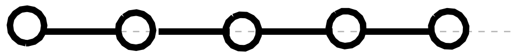
Problems that involve symmetry

Here we only consider the following particular problem.

> There are 5 identical beads connected by identical rods, and we color them with infinite supply of 2 different colors R and G.

We will discuss how to study the general situation later in weeks 7 and 8.

Example 3. How many ways are there to color these beads in the following formation.

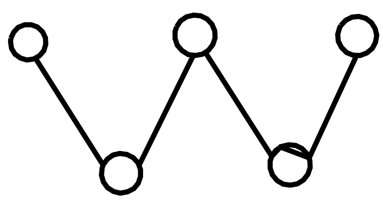


Solution. Let's say the three colors are R, G. Observe that there is the following "flip" symmetry: RRGGR and RGGRR are the same coloring. On the other hand, flipping RRGRR would cause no difference.

We carry out the coloring in the following steps.

1. Ignore the symmetry. There are 2^5 different ways doing this.
2. Divide possible colorings (ignoring symmetry) into two groups. In group A the coloring does not have "flip" symmetry, that is when "flip"ped becomes a different coloring, while in group B the coloring has "flip" symmetry. We see that the answer should be $\frac{|A|}{2} + |B| = \frac{|A| + |B|}{2} + \frac{|B|}{2} = 16 + \frac{|B|}{2}$.
3. As $|B| = 2^3$ we has the final answer to be $16 + 4 = 20$.

Exercise 3. How many ways are there to color 5 identical beads in the following formation with infinite supply of 2 colors?



Example 4. How many ways are there to color 5 identical beads connected by identical rods into a regular pentagon?

Solution. We call the two colors R and G. We have the following cases.

- All R.
  There is only 1 way to color.
- 4 R's and 1 G.
  There is only 1 way to color.
- 3 R's and 2G's.
  - The 2 G's are together. There is only 1 way to color.
  - The 2 G's are not together. Then it must be that on one side there is one R between the 2 G's and on the other side there are two R's betwen the 2 G's. There is only 1 way to color.
- 2 R's and 3 G's.
  There are 2 ways as in the 3R2G case.
- 1 R and 4 G's.
  There is only 1 way to color.
- All G.
  There is only 1 way to color.

Thus overall we have $1+1+2+2+1+1=8$ different ways to color.

Exercise 4. How many ways are there to color 6 evenly spaced identical beads on a bracelet with infinite supply of 2 colors?

Exercise 5. How many ways are there to color the four vertices of a regular tetrahedron with infinite supply of 2 colors?

Exercise 6. How many ways are there to color the vertices of a regular pentagon with infinite supply of 3 colors? (Hint: Use one color, two colors, three colors)

Finite supply of colors

Example 5. There are 7 identical beads in a box. We would like to color it with three colors R, G, B for which the supply for R is enough for 4 poles, for G 3 poles, and for B 2 poles. How many different ways are there to do the coloring?

Solution. We see that a coloring with a R's, b G's, and c B's can be identified as an integer solution of $a+b+c=7$, $0\leqslant a\leqslant 4, 0\leqslant b\leqslant 3, 0\leqslant c\leqslant 2$. The solution is

$$\binom{9}{2}-\binom{4}{2}-\binom{3}{2}-\binom{4}{2}+0+0+\binom{2}{2}+0=22. \tag{4}$$

Example 6. There are 7 different beads in a box. How many ways are there to color them with three colors R, G, B with the color R enough for 4 beads, the color G enough for 3 beads, and the color B enough the 2 beads?

Solution. We see that the answer is given by

$$\begin{aligned}\sum_{r+g+b=7,0\leqslant r\leqslant 4,0\leqslant g\leqslant 3,0\leqslant b\leqslant 2}\binom{7}{r,g,b} &= \binom{7}{4,3,0}+\binom{7}{4,2,1}\\ &\quad+\binom{7}{4,1,2}+\binom{7}{3,3,1}\\ &\quad+\binom{7}{3,2,2}+\binom{7}{2,3,2}\\ &= \frac{7!}{4!\,3!}+2\cdot\frac{7!}{4!\,2!}+\frac{7!}{(3!)^2}+2\cdot\frac{7!}{3!\,2!\,2!}\\ &= 805.\end{aligned} \tag{5}$$

Exercise 7. Convince yourself that

$$\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^5}{5!}\right)\left(1+x+\cdots+\frac{x^4}{4!}\right)\left(1+x+\cdots+\frac{x^3}{3!}\right)=\cdots+\frac{A}{7!}x^7+\cdots. \tag{6}$$