Example 1. How many different ways are there to put 100 tennis balls into 10 different boxes (assume each box is big enough for 100 tennis balls)?

Solution. We denote by x_i the number of tennis balls in the ith box. Thus the problem is equivalent to the non-negative integer solution problem for

$$x_1 + x_2 + \cdots + x_{10} = 100. \tag{1}$$

Thus the answer is $\binom{109}{9}$.

Remark 2. We see that we have treated tennis balls as identical objects.

Example 3. How many terms are there in the expansion of $(\alpha_1 + \cdots + \alpha_5)^{17}$?

Solution. A generic term is $\alpha_1^{x_1} \cdots \alpha_5^{x_5}$ with $x_i \geqslant 0$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 17$. Thus the answer is $\binom{21}{4}$.

Example 4. How many numbers between 1 and $1,000,000$ inclusive have the sum of digits 6?

Solution. Let x_i be the ith digit. Since $1,000,000$ does not have sum of digits 6, we only need to consider numbers between 1 and 999,999. Then every such number is of the form $x_1x_2x_3x_4x_5x_6$ such that $0 \leqslant x_i \leqslant 9$ and

$$x_1 + x_2 + \cdots + x_6 = 6. \tag{2}$$

The number of solutions is now given by $\binom{11}{5}$.

Exercise 1. How many numbers between 1 and $1,000,000$ inclusive have the sum of digits no more than 6?

Exercise 2. How many different ways are there to line up 8 A's and 5B's such that no two B's are adjacent? (Answer.[1])

Upper and/or lower bounds for each x_i.

- Consider the problem

  $$x_1 + \cdots + x_m = n, \qquad x_i > a_i. \tag{3}$$

  Then setting $y_i := x_i - a_i$ we see that the number of solutions is the same as the number of positive solutions of

  $$y_1 + \cdots + y_m = n - a_1 - \cdots - a_m, \tag{4}$$

  which is given by

  $$C(n - a_1 - \cdots - a_m - 1, m - 1). \tag{5}$$

  Exercise 3. How many solutions are there for

  $$x_1 + \cdots + x_m = n, \qquad x_i \geqslant a_i? \tag{6}$$

- How many solutions are there for

  $$x_1 + \cdots + x_m = n, \qquad a_i \leqslant x_i < b_i? \tag{7}$$

  We have to use inclusion-exclusion principle. We illustrate this through the following example.

  Example 5. How many ways are there to put 10 tennis balls into 3 different tubes where each tube can hold 6 balls?

  Solution. This is equivalent to solve

  $$x_1 + x_2 + x_3 = 10, \qquad 0 \leqslant x_i \leqslant 6. \tag{8}$$

  We let

  - $N_0 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_i \geqslant 0$.
  - $N_1 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 7, x_2, x_3 \geqslant 0$.
  - $N_2 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_2 \geqslant 7$, $x_1, x_3 \geqslant 0$.

1. $\binom{9}{5}$.

- $N_3 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_3 \geqslant 7$, $x_1, x_2 \geqslant 0$.
- $N_4 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1, x_2 \geqslant 7$, $x_3 \geqslant 0$.
- $N_5 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1, x_3 \geqslant 7$, $x_2 \geqslant 0$.
- $N_6 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_2, x_3 \geqslant 7$, $x_1 \geqslant 0$.
- $N_7 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1, x_2, x_3 \geqslant 7$.

Therefore we have

$$N_0 = \binom{12}{2} = 66, \quad N_1 = N_2 = N_3 = \binom{5}{2} = 10, \quad N_4 = \cdots = N_7 = 0. \tag{9}$$

Thus there are

$$66 - 30 = 36 \tag{10}$$

different ways to put the balls into the tubes.

Example 6. (COMBINATION WITH REPETITION) Consider k groups of $n_1, \ldots, n_k$ objects respectively. How many ways are there to line up m of these $n_1 + n_2 + \cdots + n_k$ objects?

We see that this is equivalent to solving

$$x_1 + \cdots + x_k = n, \qquad 0 \leqslant x_i \leqslant n_i. \tag{11}$$

Exercise 4. How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 20 \tag{12}$$

in positive integers with $x_1 \leqslant 6, x_2 \leqslant 7, x_3 \leqslant 8$, and $x_4 \leqslant 9$? (Answer:[2])

Exercise 5. How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 26 \tag{13}$$

in integers between 1 and 9 incluse? (Answer.[3])

Exercise 6. How many permutations are there of the letters of the word *Mississippi*, taken all at a time, subject to the restriction that no two i's are adjacent? (Answer:[4])

Exercise 7. Three different dice are rolled. How many outcomes have sum 10? (Answer:[5])

Exercise 8. Find the number of positive solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 25$ with the restriction that each x_i is odd.

Exercise 9. Find the number of non-negative suntions of $x_1 + \cdots + x_6 < 28$. (Hint: [6]).

Exercise 10. How many ways are there to form four blocks of four seats from 25 consecutive seats? (Hint:[7])

Exercise 11. Consider $2n$ objects among which n are identical and the other n are all different. How many ways to pick n objects from these $2n$? (Answer:[8])

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2. $\binom{19}{3} - \binom{13}{3} - \binom{12}{3} - \binom{11}{3} - \binom{10}{3} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{4}{3} + \binom{3}{3} = 217$.
3. 270.
4. 7350.
5. $\binom{9}{7} - 3\binom{3}{1}$.
6. Introduce x_7.
7. $x_1 + \cdots + x_5 = 9, x_i \geqslant 0$.
8. 2^n.