1. Permutations and combinations

The classical permutation problem considers the following situation: How many ways are there to put n distinct objects in a line?

We will study this problem and consider the following generalizations:

1. Line up only m of the n objects.
2. The n objects are not all distinct.
3. The objects are not put along a straight line, but on some other shape with symmetry.

1.1. Classical permutations.

- As the n objects are distinct, we denote them by $1, 2, ..., n$. We would like to know how many ways are there to line them up.
- Intuitively, we argue as follows: For the first number we have n choices; Once the first number is chosen, we have $n-1$ choices for the second; Once the frist two numbers are chosen, we have $n-2$ choices for the third; ... ; Once the first $n-1$ numbers are chosen, we have 1 choice for the last. Therefore $P(n,n) = n!$.
- In the above argument we have used the product rule. However it is easy to see that the successive choices are not independent of one another.

  A more mathematically rigorous argument is as follows. Let $\mathcal{C}(n)$ be the collection of all possible orderings of $\{1, 2, ..., n\}$. Then we could partition $\mathcal{C}(n) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$, where $\mathcal{C}_i$ is the collection of all orderings starting with i. It is clear that we could apply the sum rule to $\mathcal{C}$ and conclude

  $$P(n,n) = |\mathcal{C}_1| + \cdots + |\mathcal{C}_n|. \tag{1}$$

  Now let $i \in \{1, 2, ..., n\}$ be arbitrary. If we apply the bijection $1 \longleftrightarrow 1, 2 \longleftrightarrow 2, i-1 \longleftrightarrow i-1, i+1 \longleftrightarrow i$, $i+2 \longleftrightarrow i+1, ..., n \longleftrightarrow n-1$, then we see that there is a bijection $\mathcal{C}_i \longleftrightarrow \mathcal{C}(n-1)$. Thus we conclude

  $$P(n,n) = n\, P(n-1, n-1) = n\,(n-1)\, P(n-2, n-2) = \cdots = n!. \tag{2}$$

Example 1. There are 12 distinct books lined up in a shelf. If 4 are blue, how many ways are there to line them up so that all 4 blue books are together?

Solution. This could be done by a 2-step procedure. In the first step we put the 4 blue books into one case and line this case up with the other 8 books as a single book. In the second step we line up the 4 books in the case. By the product rule the answer would be the product of the number of ways to do the two steps.

For the first step we are lining up 9 distinct objects. This gives 9!. For the second step we have 4!. Therefore overall there are $9! \cdot 4!$ ways of lining these books up.

Exercise 1. In how many ways can ten persons be seated in a row so that a certain two of them are not next to each other?

Exercise 2. There are nine different books on a shelf; four are red and five are green. In how many different orders is it possible to arrange the books on the shelf if

a) there are no restrictions;

b) the red boods must be together and the green books together;

c) the red books must be together whereas the green books may be, but need not be, together;

d) the colors must alternate, i.e. no two books of the same color may be adjacent?

Exercise 3. How many different firing orders are theoretically possible in a six cylinder engine?

1.2. m-permutations.

- We consider the follow question:

How many ways are there to pick m numbers from $1, 2, ..., n$ and line them up?

- Similar to the classical permutation case, we deduce

$$P(n, m) = n \cdot (n-1) \cdot \cdots \cdot (n-m+1) = \frac{n!}{(n-m)!}. \tag{3}$$

Example 2. How many integers between 100 and 999 inclusive consist of distinct odd digits?

Solution. This is the same as the number of lists of three digits chosen from $1, 3, 5, 7, 9$ with no repitition. Thus the answer is $P(5, 3) = \frac{5!}{2!} = 60$.

Exercise 4. A room has six doors. In how many ways is it possible to enter by one and leave by another?

Exercise 5. How many integers between 138 and 976 consist of distinct odd digits?

Exercise 6. How many integers between 100 and 999 have distinct digits? Of these numbers how many are odd?

Exercise 7. How many of the first 1000 positive integers have distinct digits?

Exercise 8. How many natural numbers are there with all digits distinct?

Exercise 9. A man has a large supply of wooden regular tetrahedra, all the same size. If he paints each triangular face in one of four colors, how many different painted tetrahedra can he make, allowing all possible combinations of colors? Justify your answer.

Exercise 10. Signals are made by running five colored flags up a mast. How many different signals can be made if there is an unlimited supply of flags of seven different colors? What if adjacent flags in a signal must not be of the same color? What if all five flags in a signal must be of different colors?

1.3. Combinations and multinomial coefficients.

- We consider the following question:

  How many ways are there to line up n objects, some of them may be identical (indistinguishable)?

- The simplest case.

  Let's first consider the simplest case: All n objects are identical. In this case clearly there is only one way to line them up.

- Combinations.

  We consider the second simplest case:

  How many ways are there to line up n objects, among which m are identical and the other $n-m$ are also identical.?

  We denote the answer by $C(n, m)$. To find a formula for it we argue as follows.

  First we group the m identical objects together and mark them $1, 2, ..., m$, and mark the remaining $n-m$ identical objects $m+1, ..., n$. We know that there are $n!$ different ways lining them up. Now we erase the marks. When we do many originally "different" ways become the same. More specifically, once the positions of the m identical objects are fixed, all different "markings" of the n objects become the same. By the product rule we have $m! \cdot (n-m)!$ different ways of marking. Consequently we have

$$C(n, m) = \frac{n!}{m!\,(n-m)!} \tag{4}$$

  by applying the product rule backward.

  NOTATION 3. *An equally, if not more, popular notation for combinations is* $\binom{n}{m} := C(n, m)$. *We will use both notations in the following.*

  Remark 4. It is easy to see that $C(n, m)$ is the number of ways to choose m objects (but do not order them) from n different objects.

Example 5. A certain men's club has sixty members; thirty are business men and thirty are professors. In how many ways can a committee of eight be selected

a) if at least three must be business men and at least three professors;

b) the only condition is that at least one of the eight must be a business men?

(Leave the answers in $C(n,m)$ symbols.)

Solution.

a) There are three mutually exclusive cases:

1. 3B, 5P. In this case we have $C(30,3)\cdot C(30,5)$.
2. 4B, 4P. In this case we have $C(30,4)^2$.
3. 5B, 3P. In this case we have $C(30,5)\cdot C(30,3)$.

By the sum rule the answer is given by

$$C(30,3)\,C(30,5)+C(30,4)^2+C(30,5)\,C(30,3). \tag{5}$$

b) We first calculate that there are $C(30,8)$ different ways to form the committee without any business men, and $C(60,8)$ different ways to form the committee without any restriction. The sum rule then gives the answer as

$$C(60,8)-C(30,8). \tag{6}$$

- Multinomial coefficients.

We consider the general case:

How many ways are there to line up n objects that can be partitioned into k groups with $n_1, n_2, \ldots, n_k$ objects respectively, and the objects in the same group are identical?

Similar argument as above (for combinations) shows that

$$\binom{n}{n_1,\ldots,n_k}=\frac{n!}{n_1!\cdots n_k!} \tag{7}$$

Example 6. Consider a lattice frame in space. A bug is trying to move from $(0,0,0)$ to (n_1,n_2,n_3). The total number of different paths involving exactly $n=n_1+n_2+n_3$ steps is $\binom{n}{n_1,n_2,n_3}$.

1.4. Combinatorial identities.

Example 7. The following holds.

a) $\binom{n}{m}=\binom{n}{n-m}$.

b) $\binom{n+1}{m+1}=\binom{n}{m+1}+\binom{n}{m}$.

Proof.

a) It is clear that

$$\binom{n}{m}=\frac{n!}{m!\,(n-m)!}=\frac{n!}{(n-m)!\,m!}=\binom{n}{n-m}. \tag{8}$$

Alternatively, we have

$$\begin{aligned}\binom{n}{m} &= \text{The number of ways to choose } m \text{ objects from } n \text{ objects}\\ &= \text{The number of ways to discard } n-m \text{ objects from } n \text{ objects}\\ &= \text{The number of ways to choose } n-m \text{ objects from } n \text{ objects}\\ &= \binom{n}{n-m}.\end{aligned}$$

b) It is easy to check that

$$\begin{aligned}\binom{n+1}{m+1} &= \frac{(n+1)!}{(m+1)!\,(n-m)!}\\ &= \frac{n+1}{m+1}\frac{n!}{m!\,(n-m)!}\\ &= \frac{n-m}{m+1}\frac{n!}{m!\,(n-m)!}+\frac{n!}{m!\,(n-m)!}\\ &= \frac{n!}{(m+1)!\,(n-m-1)!}+\frac{n!}{m!\,(n-m)!}\\ &= \binom{n}{m+1}+\binom{n}{m}. \end{aligned} \tag{9}$$

Alternatively, we could give a combinatorial proof as follows.

$$\begin{aligned}\binom{n+1}{m+1} &= \text{How many ways to choose } m+1 \text{ objects from } n+1 \text{ objects}\\ &= \text{Choose the first object, and then } m \text{ from the remaining } n\\ &\quad +\text{Choose the } m+1 \text{ objects from remaining } n \text{ objects}\\ &= \binom{n}{m}+\binom{n}{m+1}.\end{aligned}$$

□

Example 8. There holds

$$\binom{n}{0}+\cdots+\binom{n}{n}=2^n. \tag{10}$$

Solution. Consider

$$(x+y)^n=(x+y)\cdot\cdots\cdot(x+y). \tag{11}$$

We see that each term in the expansion of this product is of the form $c_m\, x^m\, y^{n-m}$, where the coefficient c_m counts the number of ways to pick m x's from the n x's, which is exactly $\binom{n}{m}$. Therefore

$$(x+y)^n=\sum_{m=0}^{n}\binom{n}{m}x^m\,y^{n-m}. \tag{12}$$

Setting $x=y=1$ gives the desired result.

Exercise 11. Show that

$$(x+y+z)^n=\sum_{0\leqslant n_1,n_2,n_3;\,n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}x^{n_1}\,y^{n_2}\,z^{n_3}. \tag{13}$$

Exercise 12. Prove $\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}$.

Exercise 13. Prove $\binom{n}{0}+\binom{n+1}{1}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r}$.

Exercise 14. Prove Vandermonde's identity

$$\binom{n+m}{r}=\binom{n}{0}\binom{m}{r}+\binom{n}{1}\binom{m}{r-1}+\cdots+\binom{n}{r}\binom{m}{0}. \tag{14}$$

Exercise 15. Prove

$$\sum_{k=0}^{n}k\binom{n}{k}=2^{n-1}\cdot n. \tag{15}$$

Exercise 16. Prove

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}=4^n. \tag{16}$$

1.5. Circular permutations and beyond.

Exercise 17. (Circular permutations) In how many ways can 8 persons be seated at a round table? What if a certain two of the eight persons must not sit in adjacent seats? What if the eight persons are four men and four ladies and no two men are to be in adjacent seats? What if furthermore the persons are four married couples and no husband and wife, as well as no two men, are to be seated in adjacent seats?

Exercise 18. How many differently colored blocks of a fixed cubical shape can be made if six colors are available, and a block is to have a different color on each of its six faces?

Exercise 19. How many different cubes with the six faces numbered from 1 to 6 can be made, if the sum of the numbers on each pair of opposite faces is 7?