Product and sum rules

In the following we use $|A|$ to denote the number of elements in a collection A.

- The product rule. If the objects in the collection $\mathcal{C}$ can be obtained by making k independent choices, then

$$\mathcal{C} \longleftrightarrow A_1 \times A_2 \times A_3 \times \cdots \times A_k := \{(a_1, \ldots, a_k) | \, a_1 \in A_1, \ldots, a_k \in A_k\}. \tag{1}$$

where A_i is the set of possible ith choices. Consequently

$$|\mathcal{C}| = |A_1| \cdot \cdots \cdot |A_k|. \tag{2}$$

When to use the product rule.
1. Ordered steps (distinguishable steps); 2. Independent choices; 3. Distinguishable outcomes.

Example 1. A Canadian postal code has the format A1A 1A1. How many possible postal codes are there? The original US ZIP code has the format 11111. How many possible codes are there?

Solution. We see that for each of the first, third, and fifth positions there are 26 choices while for the second, fourth, and sixth positions there are 10 choices each. It is clear that all the choices are independent of one another. therefore there are $26^3 \times 10^3$ possible Canadian postal codes.

Similarly we see that there are 10^5 possible US ZIP codes.

Example 2. In fact, Canadian postal codes do not include the letters D, F, I, O, Q, or U, and the first position also cannot be W or Z.[1] In this case we have 18 choices for the first position, 20 for the 3rd and 5th, and 10 for each of the 2nd, 4th, and 6th. Overall there are $18 \times 10 \times 20 \times 10 \times 20 \times 10 = 7200000$ possible postal codes.

Example 3. How many $m \times n$ matrices are there with each entry either 0 or 1? What if we require there to be exactly one 1 on each row?
Solution. For each entry there are two choices 0 or 1. As there are total of $m\,n$ entries, the answer is 2^{mn}.

With the extra requirement, we model the situation differently as follows. We consider a total of n steps. At step i we choose the ith row from m choices:

$$(1, 0, \ldots, 0), \quad (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1). \tag{3}$$

Thus the total number of such matrices is given by

$$m \times m \times \cdots \times m (n \text{ times}) = m^n. \tag{4}$$

- The sum rule. If the collection $\mathcal{C}$ can be partitioned into k "subcollections" $\mathcal{C}_1, \ldots, \mathcal{C}_k$ such that every object in $\mathcal{C}$ belongs to exactly one $\mathcal{C}_i$. Further assume that each $\mathcal{C}_i$ has a bijection to some A_i, then

$$|\mathcal{C}| = |A_1| + \cdots + |A_k|. \tag{5}$$

Remark 4. The assumption "every object in $\mathcal{C}$ belongs to exactly one $\mathcal{C}_i$" is crucial. When this assumption is dropped, the sum rule does not hold anymore and we need to use the inclusion-exclusion principle which will be discussed in depth later in the course.

Example 5. How many 4-digit numbers have exactly one 8?
Solution. There are four mutually exclusive cases.

1. 8 is the thousands digit. For the other three digits we each have 9 choices so the answer is 729.

1. https://en.wikipedia.org/wiki/Postal_codes_in_Canada#Number_of_possible_postal_codes

2. 8 is the hundreds digit. For the thousands digit we have 8 choices and for the other two we have 9. So we have $8 \times 9 \times 9 = 648$.

3. 8 is the tens or ones digit. We have 648 for each.

Thus the total is

$$729 + 648 \times 3 = 2673. \tag{6}$$

Example 6. Four different dice are rolled. In how many outcomes with at least one 5 appear?
Solution. Denote by $\mathcal{C}$ the collection of outcomes with at least one 5. Let $\mathcal{C}_{\text{all}}$ be the collection of all possible outcomes of rolling four different dice, and let $\mathcal{C}_{\text{no}}$ be all possible outcomes of rolling four different dice without getting a single 5. Then we see that

$$\mathcal{C}_{\text{all}} = \mathcal{C} \cup \mathcal{C}_{\text{no}}, \qquad \mathcal{C} \cap \mathcal{C}_{\text{no}} = \varnothing. \tag{7}$$

Thus the sum rule yields

$$|\mathcal{C}_{\text{all}}| = |\mathcal{C}| + |\mathcal{C}_{\text{no}}| \Longrightarrow |\mathcal{C}| = |\mathcal{C}_{\text{all}}| - |\mathcal{C}_{\text{no}}|. \tag{8}$$

The answer is then given by $6^4 - 5^4 = 671$.

Exercise 1. Four different dice are rolled. In how many outcomes will the highest die be a 5?

Remark 7. If we consider identical dice, the sum rule cannot be applied anymore.

Example 8. How many possible 8-digit passwords are there if

a) the following symbols are allowed: $A, ..., Z, a, ..., z, 0, ..., 9$?

b) furthermore it is required that the last letter is a number?

c) instead it is required that there is at least one upper case letter and one number.

Solution.

a) For each digit we have $26 + 26 + 10 = 62$ choices. We take $A_1 = A_2 = \cdots = A_8 = \{1, 2, ..., 62\}$. Then the set of all possible passwords has a bijection with $A_1 \times \cdots \times A_8$ and therefore the answer is 62^8.

b) In this case we change A_8 to $\{1, 2, ..., 10\}$ and the answer becomes 10×62^7.

c) We calculate this through a simplest inclusion-exclusion argument.

   - Possible passwords without any requirement: 62^8;
   - Passwords with no upper case letter: 36^8.
   - Passwords without any numbers: 52^8.
   - Passwords with neither upper case letters nor numbers: 26^8.

   Therefore the answer is given by

$$62^8 - 36^8 - 52^8 + 26^8. \tag{9}$$

Example 9. Smargasbord college has four departments which have 6, 35, 12, and 7 faculty members. The president wishes to form a faculty judicial committee to hear cases of student misbehavior. To avoid possibility of ties, the committee will have three members. To avoid favoritism the committee members will be from different departments and the committee will change daily. If the committee only sits during the normal academic year (165 days), how many years can pass before a committee must be repeated? Justify your answers.
Solution. Denote the departments by A,B,C,D. There are four mutually exclusive cases, depending on which department does not contribute to the committee.

- A: $35 \times 12 \times 7$;
- B: $6 \times 12 \times 7$;

- C: $6 \times 35 \times 7$;
- D: $6 \times 35 \times 12$.

Thus the total number of possible committees is the sum of the above four numbers, which is 7434. As $\frac{7434}{165} = 45.05...$, at least 45 years would pass before the committee must be repeated.

More Exercises.

Exercise 2. The population of Carlisle, Pennsylvania, is about 20,000. If each resident has three initials, is it true that there must be at least two residents with the same initials? Give justification of your answer.

Exercise 3. How many nonnegative integers less than 1 million contain the digit 2? Justify your answer.

Exercise 4. A committee is to be chosen from among 8 scientists, 7 psychics, and 12 clerics. If the committee is to have two members of different backgrounds, how many such committees are there? Justify your answer.

Exercise 5. How many 5-letter words either start with d or do not have the letter d? Justify your answer.

Exercise 6. In how many ways can we get a sum of 3 or a sum of 4 when two dice are rolled if

a) the two dice are diffferent;

b) the two dice are identical.

Justify your answer.

Exercise 7. How many integers between 1 and 1000

a) are multiples of 13;

b) are multiples of 13 but not multiples of 7;

c) are multiples of 6 but not multiples of 9?

Justify your answer.

Exercise 8. What is the smallest number of coins (cents, nickels, dimes, quarters) needed to pay in exact change any charge less than one dollor? Justify your answer.

Exercise 9. A man has 47 cents in change coming. Assuming that the cash register contains an adequately large supply of 1, 5, 10, and 25 cent coints, with how many different combinations of coins can the clerk give the man his change? Justify your answer.

Exercise 10. How many paths are there from one corner of a cube to the opposite corner, each possible path being along three of the twelve edges of the cube? Justify your answer.

Exercise 11. How many 4-digit passwords using $A, ..., Z, 0, ..., 9$ satisfy all the following:

a) There are two letters and two numbers;

b) The letters are not adjacent;

c) If two numbers are adjacent, then they are different.

Justify your answers.

Exercise 12. A composition of a positive integer n is an ordered list of positive integers (called parts) that sum to n. The four compositions of 3 are 3; 2,1; 1,2 and 1,1,1.

a) Obtain a formula for the number of compositions of n.

b) Prove that the average number of parts in a composition of n is $(n+1)/2$. (Hint: Reverse the roles of "+" and ",").

Justify your answers.

Exercise 13. There are six high school bands lining up. How many ways are there if

a) two particular bands have to be separated; or

b) two particular bands have to be at the ends.

Exercise 14. A single die is rolled five times in a row. How many outcomes will have the fifth number equal to an earlier number?