What is counting

- When we have a collection of objects $\mathcal{C}$ and count $1, 2, 3, 4, \ldots$ what we are doing is enumerating these objects, putting them into an ordered list. Or equivalently, establishing a bijection (one-to-one and onto mapping) between the collection of objects that we need to count, and a "standard set"–a "section" of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

$$\mathcal{C} \longleftrightarrow \{1, 2, \ldots, N\}. \tag{1}$$

- For simple collections, this is easy to do. However when the collection is complicated, it becomes difficult. For example, consider the following problems:

  Problem. How many ways are there to color the faces of a cube with three colors?

  Problem. How many ways are there to list the numbers 1,2,3,...,8 so that the first, third, fifth, and seventh numbers are bigger than the numbers neighbouring them? For example 21436587 is allowed, but 87214365 is not.

  Problem. How many ways are there to tile the chessboard with 1×2 "domino"s?

  Exercise 1. Show that there are F_{n+1} ways to tile a $2 \times n$ board using 1×2 dominos, where F_n is the nth Fibonacci number: $F_1 = F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \ldots$.

- Enumerative combinatorics develops theory to solve such problems. The main idea is to find appropriate "auxiliary sets" with special structures and can be studied thoroughly. These sets then serve as a bridge between the collection to be counted and the set $\{1, 2, \ldots, N\}$.

$$\mathcal{C} \longleftrightarrow \mathcal{A} \longleftrightarrow \{1, 2, \ldots, N\}. \tag{2}$$

- Such auxiliary sets come as solutions to some "model problems". In this course we will use the following problems as model problems.

  1. Occupancy problem.

     The occupancy problem. How many different ways are there to put n balls into m cells (boxes)?

     It turns out that this leads to 8 sub-problems, depending on whether the balls or cells are distinguishable, and whether the cells are allowed to be empty.

  2. Coloring problem.

     The coloring problem. How many ways are there to color n beads with m colors?

     The basic setup is that these beads are positioned evenly along a stright line. This problem becomes complicated when the beads have some spatial structure.

  3. Integer solutions.

     Number of integer solutions. How many integer solutions are there for the equation

$$x_1 + \cdots + x_m = n? \tag{3}$$

     There are two sub-problems, depending on whether we allow x_i to be 0 or $x_i > 0$.

     Exercise 2. Show that there are infinitely many solutions to

$$x_1 + \cdots + x_m = n, \qquad x_i \in \mathbb{Z}. \tag{4}$$

     This problem can be generalized to

$$a_1 x_1 + \cdots + a_m x_m = n \tag{5}$$

     where $a_1, \ldots, a_m \in \mathbb{N}$, and further to the highly nontrivial problem of counting lattice points inside some given geometric shapes.

Remark 1. Many problems can be identified with one of the about three problems. For example,

Problem. A man works in a building located seven blocks east and eight blocks north of his home. Thus in walking to work each day he goes fifteen blocks. All the streets in the ractangular pattern are available to him for walking. In how many different ways can he go from home to work, walking only fifteen blocks?

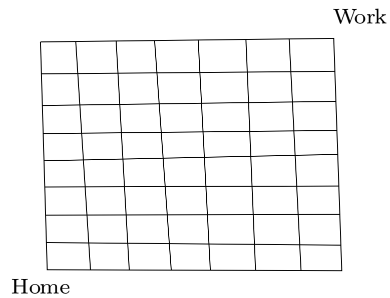

Figure 1.

Problem. A company has twenty best sellers and the governor is offered to choose ten books. He can select ten different books, or ten copies of one book, or any other combination he might prefer, provided only that the total was ten. In how many ways could the governor make his selection?

Problem. In how many ways is it possible to change a dollor bill into coins of cents, nickels, dimes, and quarters?

Exercise 3. Which model problems do the above correspond to, respectively?[1]

1. For the first problem, the man must walk 8 "up"s and 7 "right"s. It is clear that he has 9 opportunities to move right: before all the "up"s, after all the "up"s, and in between. If we denote by $x_1, \ldots, x_9$ the number of "right" moves he takes at each opportunity, then the total number of paths is given by the number of non-negative integer solutions to

$$x_1 + \cdots + x_9 = 7. \tag{6}$$