Math 421 Q1 Winter 2017 Homework 9 Solutions

Due Apr. 6, 12pm.

Total 20 points

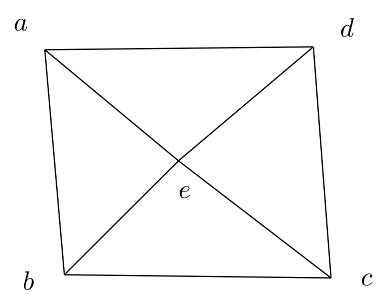
Question 1. (10 pts) *Let the graph* $G = (\{a, b, c, d, e\}, \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{e, a\}, \{e, b\}, \{e, c\}, \{e, d\}\})$.

a) (5 pts) *Draw a visualization of this graph.*

b) (5 pts) *Calculate the chromatic polynomial* $P_G(k)$. *You should simplify your polynomial to the form* $a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0$.

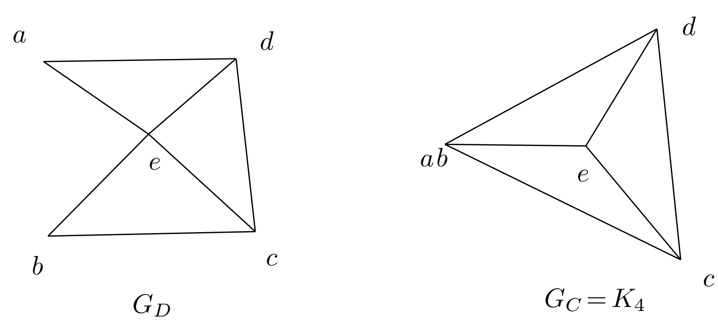
Solution.

a)



b) We apply the deletion-contraction formula repeatedly.

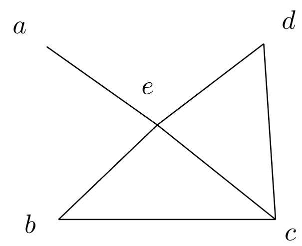
- Apply it to $\{a, b\}$ in G. We have



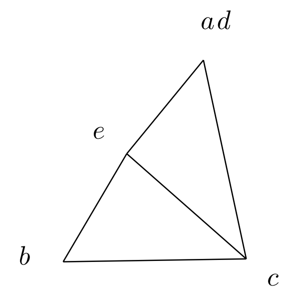
We see that

$$P_G(k) = P_{G_D}(k) - P_{K_4}(k) = P_{G_D}(k) - k\,(k-1)\,(k-2)\,(k-3). \tag{1}$$

- Apply it to $\{a, d\}$ in G_D. We have



G_{DD}

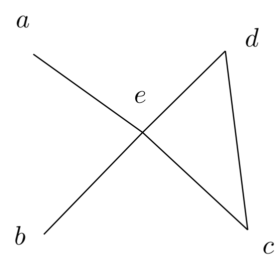


G_{DC}

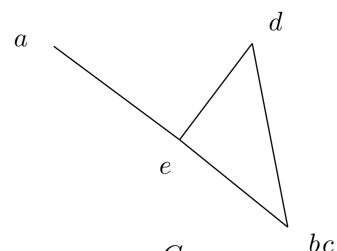
Thus we have

$$P_G(k) = P_{G_{DD}}(k) - P_{G_{DC}}(k) - k\,(k-1)\,(k-2)\,(k-3). \tag{2}$$

- $P_{G_{DD}}(k)$. We apply deletion-contraction to $\{b, c\}$, and obtain



G_{DDD}



G_{DDC}

Both are simple enough now.

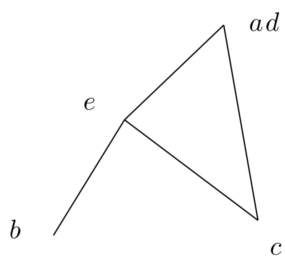
  - $P_{G_{DDD}}(k)$. We can choose any of the k colors for e. After this, there are $(k-1)$ choices for a, b, c and then $k-2$ choices for d. Therefore

$$P_{G_{DDD}}(k) = k\,(k-1)^3\,(k-2). \tag{3}$$

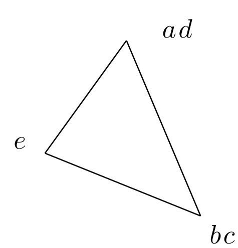
  - $P_{G_{DDC}}(k)$. We can choose any of the k colors for e. After this there are $(k-1)$ choices for a and for bc, and then $k-2$ choices for d. Therefore

$$P_{G_{DDC}}(k) = k\,(k-1)^2\,(k-2). \tag{4}$$

- $P_{G_{DC}}(k)$. We apply deletion-contraction to $\{e, b\}$ and obtain



G_{DCD}



$G_{DCC} = K_3$

We see that G_{DCD} is isomorphic to G_{DDC}, therefore

$$P_{G_{DCD}}(k) = k\,(k-1)^2\,(k-2). \tag{5}$$

On the other hand,

$$P_{G_{DCC}}(k) = P_{K_3}(k) = k\,(k-1)\,(k-2). \tag{6}$$

Putting everything together we have

$$\begin{aligned} P_G(k) &= P_{G_D}(k) - k\,(k-1)\,(k-2)\,(k-3) \\ &= P_{G_{DD}}(k) - P_{G_{DC}}(k) - k\,(k-1)\,(k-2)\,(k-3) \\ &= P_{G_{DDD}}(k) - P_{G_{DDC}}(k) - [P_{G_{DCD}}(k) - k\,(k-1)\,(k-2)] - k\,(k-1)\,(k-2)\,(k-3) \\ &= k\,(k-1)^3\,(k-2) - 2\,k\,(k-1)^2\,(k-2) + k\,(k-1)\,(k-2) - k\,(k-1)\,(k-2)\,(k-3) \\ &= k^5 - 8\,k^4 + 24\,k^3 - 31\,k^2 + 14\,k. \end{aligned} \tag{7}$$

QUESTION 2. (5 PTS) *Prove that $k^5 - k^3 + 2\,k$ cannot be a chromatic polynomial.*

Proof. Assume the contrary, that is there is a graph G such that $P_G(k) = k^5 - k^3 + 2\ k$. This gives $P_G(1) = 2 > 0$. Consequently the graph G can be colored by one single color. But this means G does not have any edges and must be a null graph, which leads to $P_G(k) = k^n$ for some $n \in \mathbb{N}$. Contradiction. □

Remark. Alternatively, $P_G(1) = 2$ means there are two ways to color the graph with one single color, which is not possible.

QUESTION 3. (5 PTS) *Prove that the coefficient of k^{n-1} in $P_G(k)$ is the negative of the number of edges. You can use the fact that for any graph of order n, its chromatic polynomial is $k^n +$ lower order terms.*

Proof. We prove by induction on the number of edges m.

- Base case. When $m = 0$, G is the null graph and we have $P_G(k) = k^n$ where the coefficient of k^{n-1} is $0 = -m$.
- Assume that for every graph with m edges, the coefficient of k^{n-1} in $P_G(k)$ is the negative of the number of edges, that is m.

  Let G be a graph of n vertices and $m + 1$ edges. Let one of the edges be $e = \{a, b\}$. We apply deletion-contraction:

  $$P_G(k) = P_{G_D}(k) - P_{G_C}(k). \tag{8}$$

  As G_D is a graph of order n with m edges, $P_{G_D}(k) = k^n - m\,k^{n-1} + \cdots$. On the other hand, as G_C is a graph of order $n-1$, there holds $P_{G_C}(k) = k^{n-1} + \cdots$. Therefore

  $$P_G(k) = k^n - (m+1)\,k^{n-1} + \cdots \tag{9}$$

Thus ends the proof. □