Math 421 Q1 Winter 2017 Homework 7 Solutions

Due Mar. 23, 12pm.

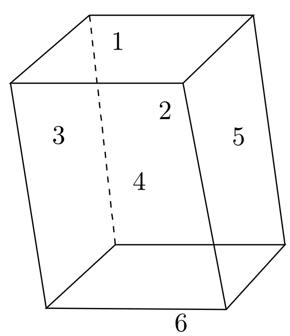
Total 20 points

Question 1. (10 pts) *Use Polya's theory to calculate the number of ways coloring the six faces of a rectangular solid that is 12 inches wide, 12 inches high, and 18 inches long, with three colors. (You do not need to justify your group G) Give your answer in numerical value.*

Solution. We first identify the group G. It has 8 elements.[1]

- The identity i.
- Three rotations of 90, 180, and 270 degrees around the line passing the centers of the two square faces. We denote them by r_{s1}, r_{s2}, r_{s3}.
- Two rotations of 180 degrees around the line passing the centers of each pair of opposite rectangular faces. We denote them by r_{r1}, r_{r2}.
- Two rotations of 180 degrees around the line passing the middle points of a pair of the opposite 18-inch edges. We denote them by r_{e1}, r_{e2}.

Now we mark the faces: Let $1, 6$ be the two square faces. Let $2, 3, 4, 5$ be the four rectangular faces, marked counter-clockwise when standing on face 1 and looking down.



Then we have

- $i = (1)(2)(3)(4)(5)(6)$. $c(i) = 6$.
- $r_{s1} = (1)(2345)(6)$. $c(r_{s1}) = 3$.
- $r_{s2} = (1)(24)(35)(6)$. $c(r_{s2}) = 4$.
- $r_{s3} = (1)(2543)(6)$. $c(r_{s3}) = 3$.
- $r_{r1} = (4)(16)(35)(2)$. $c(r_{r1}) = 4$.
- $r_{r2} = (3)(16)(24)(5)$. $c(r_{r2}) = 4$.
- $r_{e1} = (16)(45)(23)$. $c(r_{e1}) = 3$.
- $r_{e2} = (16)(34)(25)$. $c(r_{e2}) = 3$.

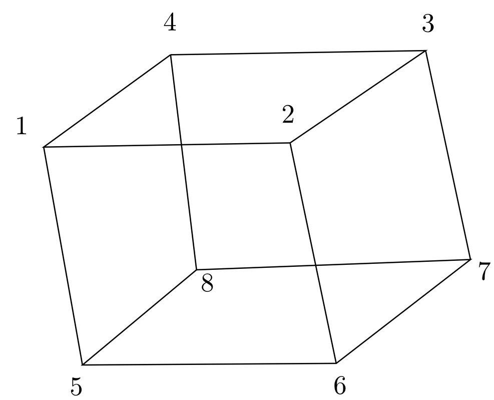
1. To see this, "embed" the rectangular solid inside the cube $[-9.9]^3$ as $[-9, 9] \times [-6, 6] \times [-6, 6]$. We see that any rotation that leaves the rectangular solid in the same position also leave the cube looking the same. Therefore the group G is a subgroup of the symmetry group of the cube. On the other hand, a rotation that leaves the cube looking the same could take the rectangular solid to $[-6, 6] \times [-9.9] \times [-6, 6]$ or $[-6, 6] \times [-6, 6] \times [-9.9]$. Therefore $|G| = 24/3 = 8$.

Therefore the answer is

$$\frac{3^6 + 4 \times 3^3 + 3 \times 3^4}{8} = 135. \tag{1}$$

QUESTION 2. (10 PTS) *Use Polya's theory to calculate the number of ways to label the vertices of a cube with either a 0 or a 1? Give your answer in numerical value.*

Solution. We see that this is the same as coloring the eight vertices with two colors. We label the eight vertices as follows.



We know that the 24 elements in G are

- The identity i. We have $c(i) = 8$.
- 6 rotations of 90 and 270 degrees around lines passing the centers of opposite faces. Consider the one of 90 degrees around the line passing the centres of the face 1234 and the face 5678. We have

  $$g = (1234)\,(5678) \Longrightarrow c(g) = 2. \tag{2}$$

  The other five rotations also have the same $c(g)$.
- 3 rotations of 180 degrees around lines passing the centers of opposite faces. Again consider the one of 90 degrees around the line passing the centres of the face 1234 and the face 5678. We have

  $$g = (13)(24)(57)(68) \Longrightarrow c(g) = 4. \tag{3}$$

  The other two rotations also have the same $c(g)$.
- 8 rotations of 120 and 240 degrees around the long diagonals. Consider the rotation of 120 degrees around the line passing vertices 1 and 7. We have

  $$g = (1)(7)(245)\,(386) \Longrightarrow c(g) = 4. \tag{4}$$

  The other 7 rotations also have the same $c(g)$.
- 6 rotations of 180 degrees around the lines passing the middle points of opposite edges. Consider the rotation around the line passing the middle points of 12 and 78. We have

  $$g = (12)(78)(35)(46) \Longrightarrow c(g) = 4. \tag{5}$$

  The other 5 rotations also have the same $c(g)$.

Thus we have

$$\text{Ans} = \frac{2^8 + 6 \times 2^2 + 17 \times 2^4}{24} = 23. \tag{6}$$