Math 421 Q1 Winter 2017 Homework 5 Solutions

Due Mar. 2, 12pm.

Total 20 points

Question 1. (5 pts) *Let a_n be the number of n-digit numbers formed by 1,3,5,7,9 with 3 and 7 appearing an even number of times. Find a formula for a_n using exponential generating functions.*

Solution. The generating function is

$$E(x) = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right) \tag{1}$$

where the five terms representing possibilities for 1,3,5,7,9 respectively. We see that

$$E(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 e^{3x} = \frac{e^{5x}}{4} + \frac{e^{3x}}{2} + \frac{e^x}{4}. \tag{2}$$

Therefore

$$a_n = \frac{1}{4}[5^n + 2 \times 3^n + 1]. \tag{3}$$

Question 2. (5 pts) *Use exponential generating function to find the number of ways color 6 pillars in a line with 4 colors (R, G, B, Y) such that the number of pillars colored R is odd and the number of pillars colored G is even.* ***Give your answer in numerical value****.*

Solution. The generating function is

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)^2 = \frac{e^x + e^{-x}}{2}\,\frac{e^x - e^{-x}}{2}\,(e^x)^2$$
$$=$$

$$= \frac{e^x + e^{-x}}{2}\,\frac{e^x - e^{-x}}{2}\,(e^x)^2 = . \tag{4}$$

It then follows that

$$a_6 = 4^5 = 1024. \tag{5}$$

Question 3. (5 pts) *Let a_n be defined through $a_{n+2} = 2\,a_{n+1} - a_n$ for all $n \geqslant 0$ and $a_0 = 0, a_1 = 1$. Use generating functions to derive the formula for a_n.*

Solution. We set $A(x) := \sum_{n=0}^{\infty} a_n\,x^n$. Then

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n\,x^n \\
&= a_0 + a_1\,x + \sum_{n=2}^{\infty} a_n\,x^n \\
&= x + \sum_{n=2}^{\infty} (2\,a_{n-1} - a_{n-2})\,x^n \\
&= x + 2\sum_{n=2}^{\infty} a_{n-1}\,x^n - \sum_{n=2}^{\infty} a_{n-2}\,x^n \\
&= x + 2\sum_{n=1}^{\infty} a_n\,x^{n+1} - \sum_{n=0}^{\infty} a_n\,x^{n+2} \\
&= x + 2\,x\,A(x) - x^2\,A(x).
\end{aligned} \tag{6}$$

This gives

$$A(x) = \frac{x}{(1-x)^2} = x \sum_{n=0}^{\infty} (n+1)\, x^n. \tag{7}$$

Consequently $a_n = n$.

QUESTION 4. (5 PTS) *Let a_n satisfy $a_{n+1} = 2\, a_n + n$ for all $n \geqslant 0$ and $a_0 = 1$. Use generating functions to find the general formula for a_n.*

Solution. Set $A(x) := \sum_{n=0}^{\infty} a_n\, x^n$. We have

$$\begin{aligned} A(x) &= a_0 + \sum_{n=1}^{\infty} a_n\, x^n \\ &= a_0 + \sum_{n=1}^{\infty} (2\, a_{n-1} + (n-1))\, x^n \\ &= a_0 + 2\, x \sum_{n=1}^{\infty} a_{n-1}\, x^{n-1} + \sum_{n=1}^{\infty} (n-1)\, x^n \\ &= a_0 + 2\, x \sum_{n=0}^{\infty} a_n\, x^n + x^2 \sum_{n=2}^{\infty} (n-1)\, x^{n-2} \\ &= a_0 + 2\, x\, A(x) + x^2 \sum_{n=0}^{\infty} (n+1)\, x^n \\ &= 1 + 2\, x\, A(x) + \frac{x^2}{(1-x)^2}. \end{aligned} \tag{8}$$

Consequently

$$A(x) = \frac{1}{1-2\,x} + \frac{x^2}{(1-2\,x)\,(1-x)^2} = \frac{2}{1-2\,x} - \frac{1}{(1-x)^2}. \tag{9}$$

Therefore

$$a_n = 2^{n+1} - n - 1. \tag{10}$$