Math 421 Winter 2017 Midterm 1 Solutions

Feb. 3, 2017 1pm - 1:50pm. Total 30 Pts

NAME: ID#:

- Please write clearly and **show enough work** (this is very important!).
- No electronic devices are allowed.

Question 1. (5 pts) *A single die is rolled four times in a row. How many outcomes will have all four numbers different? You should calculate the numerical value of the answer.*

Solution. This is the same as lining up 4 different numbers from $1, 2, \ldots, 6$ so it is $P(6, 4) = 6 \times 5 \times 4 \times 3 = 360$.

Remark. Rolling a single die 4 times is the same as rolling 4 different dice once, but different from rolling 4 identical dice once. As when a single die is rolled 4 times, there is a difference between getting a 1 followed by three 2's, and getting three 2's followed by a 1. On the other hand, when 4 identical dice are rolled once, there is just one possible "three 2's and one 1" outcome.

One may argue that in the former case (one die rolled 4 times) the observer can choose to "ignore" the order. That is true. But note that in the latter case the oberver cannot choose to "establish" an order–if the observer could do so, the dice are not identical.

Question 2. (5 pts) *How many ways are there to put 6 different balls into 4 different boxes with no box left empty? Give your answer in numerical value.*

Solution. This is standard situation.

$$T(6, 4) = 4^6 - 4 \times 3^6 + 6 \times 2^6 - 4 = 4096 - 2916 + 384 - 4 = 1560. \quad (1)$$

Question 3. (5 pts) *How many ways are there to put 8 different balls into 3 identical boxes with no box left empty? Give your answer in numerical value.*

Solution. This is standard situation.

$$S(8,3) = \frac{1}{3!} T(8,3) = \frac{3^8 - 3 \times 2^8 + 3}{6} = \frac{6561 - 768 + 3}{6} = 966. \tag{2}$$

Question 4. (5 pts) *How many integer solutions are there to*

$$x_1 + x_2 + x_3 = 11, \qquad 1 \leqslant x_1 \leqslant 10,\ 3 \leqslant x_2 \leqslant 9,\ -1 \leqslant x_3 \leqslant 5? \tag{3}$$

Calculate the numerical value of the answer.

Solution. We set

- A_0=solutions to $x_1 + x_2 + x_3 = 11$, $x_1 \geqslant 1$, $x_2 \geqslant 3$, $x_3 \geqslant -1$. We have $|A_0| = \binom{10}{2} = \frac{10!}{2!\,8!} = \frac{10 \times 9}{2 \times 1} = 45$.
- A_1 = solutions to $x_1 + x_2 + x_3 = 11$, $x_1 \geqslant 11$, $x_2 \geqslant 3$, $x_3 \geqslant -1$. We have $|A_1| = 0$.
- A_2=solutions to $x_1 + x_2 + x_3 = 11$, $x_1 \geqslant 1$, $x_2 \geqslant 10$, $x_3 \geqslant -1$. We have $|A_2| = \binom{3}{2} = 3$.
- A_3 = solutions to $x_1 + x_2 + x_3 = 11$, $x_1 \geqslant 1$, $x_2 \geqslant 3$, $x_3 \geqslant 6$. We have $|A_3| = \binom{3}{2} = 3$.
- $A_1 \cap A_2$ = solutions to $x_1 + x_2 + x_3 = 11, x_1 \geqslant 11, x_2 \geqslant 3, x_3 \geqslant 6$. We have $|A_1 \cap A_2| = 0$.
- $A_2 \cap A_3$=solutions to $x_1 + x_2 + x_3 = 11, x_1 \geqslant 1, x_2 \geqslant 10, x_3 \geqslant 6$. We have $|A_2 \cap A_3| = 0$.
- $A_1 \cap A_3$=solutions to $x_1 + x_2 + x_3 = 11, x_1 \geqslant 11, x_2 \geqslant 10, x_3 \geqslant -1$. We have $|A_1 \cap A_3| = 0$.
- $A_1 \cap A_2 \cap A_3$ = solutions to $x_1 + x_2 + x_3 = 11, x_1 \geqslant 11, x_2 \geqslant 3, x_3 \geqslant 6$. We have $|A_1 \cap A_2 \cap A_3| = 0$.

We have the answer to be

$$|A_0| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| = 39. \tag{4}$$

Question 5. (5 pts) *How many different terms are there in the expansion of*

$$(x + y + z)^{10} (t + u + v + w)^4 (p + q + r + s)^3? \tag{5}$$

Calculate the numerical value of your answer.

Solution. A generic term in $(x+y+z)^{10}$ is of the form $x^a\, y^b\, z^c$ with $a+b+c=10$, $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$. On the other hand, for every triplet a, b, c satisfying $a+b+c=10$, clearly $x^a\, y^b\, z^c$ appears in the expansion of $(x+y+z)^{10}$. Therefore the number of terms in the expansion of $(x+y+z)^{10}$ is the same as the number of integer solutions to

$$a+b+c=10, \qquad a,b,c \geqslant 0, \tag{6}$$

which is $\binom{12}{2}=66$.

Similarly, there are $\binom{7}{3}=35$ terms in the expansion of $(t+u+v+w)^4$ and $\binom{6}{3}=20$ terms in the expansion of $(p+q+r+s)^3$.

A generic term in (5) can be obtained through a 3-step process:

1. Pick one term from $(x+y+z)^{10}$;
2. Multiply by one term from $(t+u+v+w)^4$;
3. Multiply by one term from $(p+q+r+s)^3$.

The three steps are clearly independent. Therefore the total number of different terms in (5) is now given by the product rule as is $66 \times 35 \times 20 = 46200$.

Question 6. (5 pts) *How many different ways are there to color a regular tetrahedron with three colors?*

Solution. We divide into the following cases.

- Use one color only. There are 3 ways.
- Use two colors. Then there are two sub-situations.
    - One color for one face, the other color for the remaining three faces. Due to symmetry, we see that we can always rotate the tetrahedron that the "one color" face faces downward. Thus once the color for the "one face" and the color for the "three faces" are fixed, there is just one way to color the tetrahedron. As there are $P(3,2)=6$ different ways picking these two colors (color one face red and three other faces green is clearly different from color one face green and three faces red), there are in total 6 ways to color one face with a color and three faces with another color.

  - Color two faces with one color and two other faces with another color. Say the colors are R and G. Due to the symmetry, we can always rotate the tetrahedron that the two "R" faces are facing downward and forward. We see that there is only one ways to color two faces with R and two faces with G. There are $C(3,2)=3$ different ways to pick two colors out of three. So there are in total 3 ways to color two faces with a color and two faces with another color.

  So there are $6+3=9$ different ways to color the tetrahedron with two colors.

- Use all three colors.

  In this case one color is used on two faces and the other two color each for one face. Let's fix the colors first. Say we use R for two faces and G,B each for one face. Due to symmetry we can always make the two "R" faces facing forward and downward. Say now we have G facing left and B facing right. Now if we "flip" the tetrehedron, we see that the "R" faces are still facing forward and downward, but now G is facing right and B is facing left. Therefore there is only one way to color two faces with R and one face with G, B. Now there are three different ways to choose the "two-face" color, so there are 3 different ways to color the tetrahedron with all three colors.

Thus overall we have $3+9+3=15$ different ways.

Remark. One may notice that the answer is exactly the number of integer solutions to

$$r+g+b=4, \qquad r\geqslant 0, g\geqslant 0, b\geqslant 0. \tag{7}$$

That is, the number of different ways to color a regular tetrahedron with three different colors, is the same as the number of different ways to put four identical balls into three different boxes.

Is this a coincidence? Try figure this out through working on the problem of coloring a regular tetrahedron with two different colors, or four different colors. We will discuss more about this in weeks seven and eight.