Math 421 Q1 Winter 2017 Homework 3 Solutions

Due Feb. 9, 12pm.

Total 20 points.

Question 1. (5 pts) *How many ways are there to put 8 different books into 3 different boxes such that no box is left empty? The answer should be given in numerical value.*

Solution. The answer is

$$\begin{aligned} T(8,3) &= 3^8 - 3\times 2^8 + 3\times 1^8 \\ &= 6561 - 768 + 3 \\ &= 5796. \end{aligned} \tag{1}$$

Question 2. (5 pts) *How many ways are there to put 8 different books into 3 identical boxes (the boxes can be empty)? The answer should be given in numerical value.*

Solution. The answer is

$$\begin{aligned} S(8,1)+S(8,2)+S(8,3) &= 1 + \frac{T(8,2)}{2} + \frac{T(8,3)}{6} \\ &= 1 + \frac{2^8-2}{2} + \frac{5796}{6} \\ &= 1 + 127 + 966 \\ &= 1094. \end{aligned} \tag{2}$$

Question 3. (5 pts) *6 identical bottles of Pepsi and 6 identical bottles of Crush are distributed to three people. How many ways are there to do this if each person receives at least one bottle (doesn't matter Pepsi or Crush)? The answer should be given in numerical value.*

Solution. First we ignore the requirement that each person receives at least one bottle. We can distribute the bottles in two steps.

1. Distribute the 6 bottles of Pepsi. This is the same as the number of solutions to

$$x_1 + x_2 + x_3 = 6, \qquad x_i \geqslant 0 \tag{3}$$

   which is $C(8,2) = 28$.

2. Distribute the 6 bottles of Crush. There are again 28 ways.

Thus the total is $28^2 = 784$.

Now we subtract from 784 the number of ways of distributing these bottles with at least one person left empty-handed. We have:

- N_1: First person empty handed. Doing the two-step procedure as above, we see that each step is the same as $x_2 + x_3 = 6, x_2, x_3 \geqslant 0$. $N_1 = 7 \times 7 = 49$.
- N_2: Second person empty handed. $N_2 = N_1 = 49$.
- N_3: Third person empty handed. $N_3 = N_2 = N_1 = 49$.
- N_4: First and second person empty handed. Clearly $N_4 = 1$.
- N_5: First and third person empty handed. $N_5 = 1$.
- N_6: Second and third person empty handed. $N_6 = 1$.
- N_7: All three people empty handed. $N_7 = 0$.

Thus the answer is

$$784 - 3 \times 49 + 3 \times 1 = 640. \tag{4}$$

QUESTION 4. (5 PTS) *Prove that the number of partitions of n into three positive summands is equal to the number of partitions of $2n$ into three positive summands of size less than n.*

Proof. We consider the following correspondence: For any $a + b + c = 2n$ with $a, b, c \leqslant n$, define $a' = n - a$, $b' = n - b$, $c' = n - c$. Then we have $a' + b' + c' = n$, $a', b', c' \geqslant 0$. We claim that this gives a bijection between partitions of of n into three summands and partitions of $2n$ into three summands of size less than n.

- For every partition a', b', c' of n, we have $a = n - a', b = n - b', c = n - c'$ to be a partition of $2n$ and furthermore $a, b, c < n$.
- If a', b', c' and d', e', f' are different partitions of n, clearly $a = n - a', b = n - b', c = n - c'$ and $d = n - d'$, $e = n - e'$, $f = n - f'$ are different partitions with summands less than n.
- If a, b, c is a partition of $2n$ with summands less than n, $a' = n - a, b' = n - b, c' = n - c$ would satisfy $a + b + c = n$, $a, b, c > 0$. Thus a, b, c is a partition of n into three summands.

Therefore $(a', b', c') \longleftrightarrow (a = n - a', b = n - b', c = n - c')$ is a bijection between partitions of of n into three summands and partitions of $2n$ into three summands of size less than n. As a consequence the two collections have the same number of partitions. □