# Math 421 Q1 Winter 2017 Homework 2 Solutions

Due Jan. 26, 12pm.

Total 20 points.

Question 1. (5 pts) *How many ways are there to line up 1,2,...,9 such that 1 is somewhere to the right of 2 and 2 is somewhere to the right of 3? Justify your answer.*

**Solution 1.** Note that if for every such line up we freely re-order $1,2,3$, then we obtain all permutations of $1,2,...,9$. Consequently the answer is $\frac{9!}{3!} = 60480$.

**Solution 2.** There are 6! different ways to line up $4,5,6,....,9$. Once this is done, we insert $3,2,1$ from left to right. If we set $x_1$ to be the number of numbers to the left of 3, $x_2$ to be the number of numbers between 3 and 2, $x_3$ to be the number of numbers between 2 and 1, and $x_4$ to be the number of numbers to the right of 1, then the number of different ways of inserting $3,2,1$ is the same as the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 6, \qquad x_i \geqslant 0 \tag{1}$$

which is $\binom{9}{3}$. So the final answer is $\binom{9}{3} \cdot 6! = \frac{9!}{6} = 60480$.

**Solution 3.** We line up $1,2,...,9$, that is, putting numbers $1,2,...,9$ into positions $1,2,...9$, in three steps.

1. Choose 6 positions from the 9 positions. There are $\binom{9}{6}$ different ways.
2. Putting $4,...,9$ into these 9 positions. There are 6! different ways.
3. Putting $1,2,3$ into the remaining 3 positions. There is one way to do this.

So the answer is given by the product rule: $\binom{9}{6} \cdot 6! \cdot 1 = \frac{9!}{3!} = 60480$.

**Note.** (to the Grader) Getting the number 60480 is not required.

Question 2. (5 pts) *How many ways are there to form four blocks of four seats from 25 consecutive seats? Justify your answer. (Here as the seats are already put in a line, they are seen as different: 1st seat, 2nd seat, 3rd seat, ..., and the blocks cannot overlap.)*

**Solution.** Let $x_1$ be the number of seats before the first block, $x_2$ be the number of seats between the first and the second blocks, $x_3$ be the number of seats between the second and the third blocks, $x_4$ be the number of seats between the third and fourth blocks, and $x_5$ be the number of seats after the fourth block of seats. Then we see that the answer is the same as the number of solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25 - 4 \times 4 = 9, \qquad x_i \geqslant 0, i = 1,2,3,4,5, \tag{2}$$

which is

$$\binom{9+5-1}{5-1} = \binom{13}{4} = 715. \tag{3}$$

**Note.** (to the Grader) Getting the number 715 is not required.

Question 3. (5 pts) *How many solutions are there for*

$$x_1 + x_2 + x_3 = 10, \qquad 0 \leqslant x_1 < 5, \quad -1 \leqslant x_2 < 6, \quad 2 \leqslant x_3 < 7? \tag{4}$$

*Justify your answer.*

**Solution.** Let's define the following numbers:

- $N_0 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 0, x_2 \geqslant -1, x_3 \geqslant 2$. Setting $y_1 = x_1$, $y_2 = x_2 + 1$, $y_3 = x_3 - 2$ we see that

$$y_1 + y_2 + y_3 = 9, \qquad y_i \geqslant 0 \tag{5}$$

so

$$N_0 = \binom{11}{2} = 55. \tag{6}$$

- $N_1 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 5, x_2 \geqslant -1, x_3 \geqslant 2$. Setting $y_1 = x_1 - 5, y_2 = x_2 + 1$, $y_3 = x_3 - 2$ we see that

$$y_1 + y_2 + y_3 = 4, \qquad y_i \geqslant 0 \tag{7}$$

so

$$N_1 = \binom{6}{2} = 15. \tag{8}$$

- $N_2 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 0$, $x_2 \geqslant 6$, $x_3 \geqslant 2$. We have

$$N_2 = \binom{4}{2} = 6. \tag{9}$$

- $N_3 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 0$, $x_2 \geqslant -1$, $x_3 \geqslant 7$. We have

$$N_3 = \binom{6}{2} = 15. \tag{10}$$

- $N_4 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 5$, $x_2 \geqslant 6$, $x_3 \geqslant 2$. We have

$$N_4 = 0. \tag{11}$$

- $N_5 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 5$, $x_3 \geqslant 7$, $x_2 \geqslant -1$. We have

$$N_5 = 0. \tag{12}$$

- $N_6 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 0$, $x_2 \geqslant 6$, $x_3 \geqslant 7$. We have

$$N_6 = 0. \tag{13}$$

- $N_7 :=$ the number of solutions to $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 5$, $x_2 \geqslant 6$, $x_3 \geqslant 7$. We have

$$N_7 = 0. \tag{14}$$

Then the answer is

$$N_0 - (N_1 + N_2 + N_3 - N_4 - N_5 - N_6 + N_7) = 19. \tag{15}$$

**Note.** (TO THE GRADER) It is OK to leave $N_0, \ldots, N_3$ are combination numbers.

QUESTION 4. (5 PTS) *A bag of coins contains eight nickles, four dimes, and three quarters. Assuming that coins of any one denomination are identical, in how many ways can a collection of ten coins be made up from the bagful? Justify your answer.*

**Solution.** The answer is the same as the number of integer solutions to

$$x_1 + x_2 + x_3 = 10, \qquad 0 \leqslant x_1 \leqslant 8, \quad 0 \leqslant x_2 \leqslant 4, \quad 0 \leqslant x_3 \leqslant 3. \tag{16}$$

Setting $N_0, \ldots, N_7$ to be the number of solutions to the following problems:

- $N_0$: $x_1 + x_2 + x_3 = 10$, $x_1, x_2, x_3 \geqslant 0$; Thus $N_0 = \binom{12}{2} = 66$.
- $N_1$: $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 9$, $x_2, x_3 \geqslant 0$; Thus $N_1 = \binom{3}{2} = 3$.
- $N_2$: $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 0$, $x_2 \geqslant 5$, $x_3 \geqslant 0$; Thus $N_2 = \binom{7}{2} = 21$.
- $N_3$: $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 4$; Thus $N_3 = \binom{8}{2} = 28$.
- $N_4$: $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 9$, $x_2 \geqslant 5$, $x_3 \geqslant 0$; Thus $N_4 = 0$.
- $N_5$: $x_1 + x_2 + x_3 = 10$, $x_1 \geqslant 9$, $x_2 \geqslant 0$, $x_3 \geqslant 4$; Thus $N_5 = 0$.

- $N_6$: $x_1+x_2+x_3=10,\ x_1\geqslant 0,\ x_2\geqslant 5,\ x_3\geqslant 4$; Thus $N_6=\binom{3}{2}=3$.
- $N_7$: $x_1+x_2+x_3=10,\ x_1\geqslant 9,\ x_2\geqslant 4,\ x_3\geqslant 4$; Thus $N_7=0$.

Finally we have the answer to be

$$N_0-[N_1+N_2+N_3-N_4-N_5-N_6+N_7]=17. \tag{17}$$

**Note.** (TO THE GRADER) It is OK to leave $N_0, ..., N_3$ are combination numbers.