LECTURES 16–17: GAUSS'S REMARKABLE THEOREM

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

Much of the material in this lecture is optional. On the other hand, it is beneficial to work through the notes as the calculations etc. here can serve as good review of the concepts/formulas we studied in the past two months.

The required textbook sections are §10.1–10.2. The optional sections are §10.3–10.4.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Gauss' Remarkable Theorem

1.1. Motivation

We have seen that given a surface, one can calculate its first and second fundamental forms

$$\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2 \text{ and } \mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2, \tag{1}$$

as well as the Christoffel symbols Γ_{ij}^k, defined through

$$\begin{aligned} \sigma_{uu} &= \Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N, \\ \sigma_{uv} &= \Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N, \\ \sigma_{vv} &= \Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N. \end{aligned} \tag{2}$$

We have also seen that the Christoffel symbols are not independent quantities and can be calculated from $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$. Now the question is, are $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{L}$, $\mathbb{M}$, $\mathbb{N}$ independent? In other words, given six functions $\mathbb{E}(u,v)$, ..., $\mathbb{N}(u,v)$, is there always a surface S having (1) as its first and second fundamental forms?

For example, is there a surface with first and second fundamental forms $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$ and $\cos^2 u\,\mathrm{d}u^2 + \mathrm{d}v^2$? To answer this, we need to first understand whether $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ are related.

1.2. Codazzi-Mainradi equations and Gauss equations

THEOREM 1. *Let S be a surface and let $\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2$, $\mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2$, and Γ_{ij}^k be its first, second fundamental forms, and Christoffel symbols. Then there hold*

- *the Codazzi-Mainradi equations*

$$\begin{aligned} \mathbb{L}_v - \mathbb{M}_u &= \mathbb{L}\,\Gamma_{12}^1 + \mathbb{M}\,(\Gamma_{12}^2 - \Gamma_{11}^1) - \mathbb{N}\,\Gamma_{11}^2 \\ \mathbb{M}_v - \mathbb{N}_u &= \mathbb{L}\,\Gamma_{22}^1 + \mathbb{M}\,(\Gamma_{22}^2 - \Gamma_{12}^1) - \mathbb{N}\,\Gamma_{12}^2 \end{aligned}, \tag{3}$$

- *and the Gauss equations*

$$\begin{aligned} \mathbb{E}\,K &= (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\,\Gamma_{12}^2 + \Gamma_{11}^2\,\Gamma_{22}^2 - \Gamma_{12}^1\,\Gamma_{11}^2 - (\Gamma_{12}^2)^2, \\ \mathbb{F}\,K &= (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\,\Gamma_{12}^1 - \Gamma_{11}^2\,\Gamma_{22}^1, \\ &= (\Gamma_{12}^2)_v - (\Gamma_{22}^2)_u + \Gamma_{12}^1\,\Gamma_{12}^2 - \Gamma_{22}^1\,\Gamma_{11}^2, \\ \mathbb{G}\,K &= (\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1\,\Gamma_{11}^1 + \Gamma_{22}^2\,\Gamma_{12}^1 - (\Gamma_{12}^1)^2 - \Gamma_{12}^2\,\Gamma_{22}^1. \end{aligned} \tag{4}$$

Proof. By (2) we have

$$(\Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N)_v = \sigma_{uuv} = \sigma_{uvu} = (\Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N)_u \tag{5}$$

and

$$(\Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N)_v = \sigma_{uvv} = \sigma_{vvu} = (\Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N)_u. \tag{6}$$

Now calculate

$$\begin{aligned}(\Gamma_{11}^1\,\sigma_u+\Gamma_{11}^2\,\sigma_v+\mathbb{L}\,N)_v &= (\Gamma_{11}^1)_v\,\sigma_u+\Gamma_{11}^1\,\sigma_{uv}+(\Gamma_{11}^2)_v\,\sigma_v+\Gamma_{11}^2\,\sigma_{vv}+\mathbb{L}_v\,N+\mathbb{L}\,N_v\\ &= (\Gamma_{11}^1)_v\,\sigma_u+(\Gamma_{11}^2)_v\,\sigma_v+\mathbb{L}_v\,N\\ &\quad+\Gamma_{11}^1\,(\Gamma_{12}^1\,\sigma_u+\Gamma_{12}^2\,\sigma_v+\mathbb{M}\,N)\\ &\quad+\Gamma_{11}^2\,(\Gamma_{22}^1\,\sigma_u+\Gamma_{22}^2\,\sigma_v+\mathbb{N}\,N)\\ &\quad+\mathbb{L}\,(-a_{21}\,\sigma_u-a_{22}\,\sigma_v)\\ &= [(\Gamma_{11}^1)_v+\Gamma_{11}^1\,\Gamma_{12}^1+\Gamma_{11}^2\,\Gamma_{22}^1-a_{21}\,\mathbb{L}]\,\sigma_u\\ &\quad+[(\Gamma_{11}^2)_v+\Gamma_{11}^1\,\Gamma_{12}^2+\Gamma_{11}^2\,\Gamma_{22}^2-a_{22}\,\mathbb{L}]\,\sigma_v\\ &\quad+[\mathbb{L}_v+\Gamma_{11}^1\,\mathbb{M}+\Gamma_{11}^2\,\mathbb{N}]\,N. \end{aligned} \tag{7}$$

Here recall that $\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}=\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}$.

Similarly we calculate

$$\begin{aligned}(\Gamma_{12}^1\,\sigma_u+\Gamma_{12}^2\,\sigma_v+\mathbb{M}\,N)_u &= (\Gamma_{12}^1)_u\,\sigma_u+\Gamma_{12}^1\,\sigma_{uu}+(\Gamma_{12}^2)_u\,\sigma_v+\Gamma_{12}^2\,\sigma_{uv}+\mathbb{M}_u\,N+\mathbb{M}\,N_u\\ &= (\Gamma_{12}^1)_u\,\sigma_u+(\Gamma_{12}^2)_u\,\sigma_v+\mathbb{M}_u\,N\\ &\quad+\Gamma_{12}^1\,(\Gamma_{11}^1\,\sigma_u+\Gamma_{11}^2\,\sigma_v+\mathbb{L}\,N)\\ &\quad+\Gamma_{12}^2\,(\Gamma_{12}^1\,\sigma_u+\Gamma_{12}^2\,\sigma_v+\mathbb{M}\,N)\\ &\quad+\mathbb{M}\,(-a_{11}\,\sigma_u-a_{12}\,\sigma_v)\\ &= [(\Gamma_{12}^1)_u+\Gamma_{12}^1\,\Gamma_{11}^1+\Gamma_{12}^2\,\Gamma_{12}^1-a_{11}\,\mathbb{M}]\,\sigma_u\\ &\quad+[(\Gamma_{12}^2)_u+\Gamma_{12}^1\,\Gamma_{11}^2+\Gamma_{12}^2\,\Gamma_{12}^2-a_{12}\,\mathbb{M}]\,\sigma_v\\ &\quad+[\mathbb{M}_u+\Gamma_{12}^1\,\mathbb{L}+\Gamma_{12}^2\,\mathbb{M}]\,N. \end{aligned} \tag{8}$$

As $\{\sigma_u,\sigma_v,N\}$ form a basis of $\mathbb{R}^3$, there must hold

$$(\Gamma_{11}^1)_v+\Gamma_{11}^1\,\Gamma_{12}^1+\Gamma_{11}^2\,\Gamma_{22}^1-a_{21}\,\mathbb{L} = (\Gamma_{12}^1)_u+\Gamma_{12}^1\,\Gamma_{11}^1+\Gamma_{12}^2\,\Gamma_{12}^1-a_{11}\,\mathbb{M}, \tag{9}$$

$$(\Gamma_{11}^2)_v+\Gamma_{11}^1\,\Gamma_{12}^2+\Gamma_{11}^2\,\Gamma_{22}^2-a_{22}\,\mathbb{L} = (\Gamma_{12}^2)_u+\Gamma_{12}^1\,\Gamma_{11}^2+\Gamma_{12}^2\,\Gamma_{12}^2-a_{12}\,\mathbb{M}, \tag{10}$$

$$\mathbb{L}_v+\Gamma_{11}^1\,\mathbb{M}+\Gamma_{11}^2\,\mathbb{N} = \mathbb{M}_u+\Gamma_{12}^1\,\mathbb{L}+\Gamma_{12}^2\,\mathbb{M}. \tag{11}$$

We see that (11) immediately gives the first Codazzi-Mainradi equation. (9) yields

$$a_{11}\,\mathbb{M}-a_{21}\,\mathbb{L}=(\Gamma_{12}^1)_u-(\Gamma_{11}^1)_v+\Gamma_{12}^2\,\Gamma_{12}^1-\Gamma_{11}^2\,\Gamma_{22}^1, \tag{12}$$

which becomes the second Gauss equation after we notice the following.

$$\begin{aligned}a_{11}\,\mathbb{M}-a_{21}\,\mathbb{L} &= -\left[\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}\mathbb{N} & -\mathbb{M}\\ -\mathbb{M} & \mathbb{L}\end{pmatrix}\right]_{(12)}\\ &= -\left[\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}\begin{pmatrix}\mathbb{N} & -\mathbb{M}\\ -\mathbb{M} & \mathbb{L}\end{pmatrix}\right]_{(12)}\\ &= -(\mathbb{L}\,\mathbb{N}-\mathbb{M}^2)\left[\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\right]_{(12)}\\ &= -(\mathbb{L}\,\mathbb{N}-\mathbb{M}^2)\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)^{-1}\begin{pmatrix}\mathbb{G} & -\mathbb{F}\\ -\mathbb{F} & \mathbb{E}\end{pmatrix}_{(12)}\\ &= \frac{\mathbb{L}\,\mathbb{N}-\mathbb{M}^2}{\mathbb{E}\,\mathbb{G}-\mathbb{F}^2}\,\mathbb{F}=\mathbb{F}\,K. \end{aligned} \tag{13}$$

Here we use $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(ij)}$ to denote the (i, j) component of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Similar calculations confirm the rest of the equations. $\square$

Remark 2. As Γ_{ij}^k can be calculated using $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$, the Codazzi-Mainradi and Gauss equations can be seen as equations for $\mathbb{L}, \mathbb{M}, \mathbb{N}$ given $\mathbb{E}, \mathbb{F}, \mathbb{G}$.

Remark 3. It can be shown that as long as $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ satisfy the Codazzi-Mainradi and Gauss equations, with Γ_{ij}^k calculated from $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ as we have seen before, and with $K := \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}$, then there is a unique surface patch σ with $\mathbb{E}\ \mathrm{d}u^2 + 2\ \mathbb{F}\ \mathrm{d}u\ \mathrm{d}v + \mathbb{G}\ \mathrm{d}v^2$, $\mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2$ as its first and second fundamental forms, Γ_{ij}^k as its Christoffel symbols, and K as its Gaussian curvature.[1]

Example 4. [2]Is there a surface with the first and second fundamental forms $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$ and $\cos^2 u\,\mathrm{d}u^2 + \mathrm{d}v^2$?

We see that $\mathbb{E} = 1, \mathbb{F} = 0, \mathbb{G} = \cos^2 u, \mathbb{L} = \cos^2 u, \mathbb{M} = 0, \mathbb{N} = 1$. Now we calculate Γ_{ij}^k. As it is too hard to remember the formulas, we start from the "first principle" (2):

$$\begin{aligned} \sigma_{uu} &= \Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N, \\ \sigma_{uv} &= \Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N, \\ \sigma_{vv} &= \Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N. \end{aligned} \tag{14}$$

We have

$$\Gamma_{11}^1 = \Gamma_{11}^1\,\mathbb{E} + \Gamma_{11}^2\,\mathbb{F} = \sigma_{uu} \cdot \sigma_u = \frac{1}{2}\,\mathbb{E}_u = 0. \tag{15}$$

$$\cos^2 u\,\Gamma_{11}^2 = \Gamma_{11}^1\,\mathbb{F} + \Gamma_{11}^2\,\mathbb{G} = \sigma_{uu} \cdot \sigma_v = (\sigma_u \cdot \sigma_v)_u - \frac{1}{2}\,(\sigma_u \cdot \sigma_u)_v = 0 \tag{16}$$

Thus $\Gamma_{11}^1 = \Gamma_{11}^2 = 0$. Similarly we can calculate

$$\Gamma_{12}^1 = 0, \qquad \Gamma_{12}^2 = -\tan u, \qquad \Gamma_{22}^1 = \cos u \sin u, \qquad \Gamma_{22}^2 = 0. \tag{17}$$

Now we see that

$$\mathbb{M}_v - \mathbb{N}_u = 0 \tag{18}$$

while

$$\mathbb{L}\,\Gamma_{22}^1 + \mathbb{M}\,(\Gamma_{22}^2 - \Gamma_{12}^1) - \mathbb{N}\,\Gamma_{12}^2 = \cos^3 u \sin u + \tan u \neq 0. \tag{19}$$

Thus the second Codazzi-Mainradi equation is not satisfied and consequently there is no surface with the first and second fundamental forms $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$ and $\cos^2 u\,\mathrm{d}u^2 + \mathrm{d}v^2$.

1.3. Gauss' remarkable theorem

THEOREM 5. (GAUSS' THEOREMA EGREGIUM) *The Gaussian curvature of a surface is preserved by local isometries.*

Proof. This follows immediately from the Gauss equations together with the fact that if $f\colon S \mapsto \tilde{S}$ is a local isometry, then the first fundamental forms for $\sigma(u, v)$ and $\tilde{\sigma}(u, v) := f(\sigma(u, v))$ are identical, and consequently the two surfaces have the same $\mathbb{E}, \mathbb{F}, \mathbb{G}, \Gamma_{ij}^k$. $\square$

1. Theorem 10.1.3 of the textbook.
2. Exercise 10.1.2 of the textbook.

Remark 6. One can calculate an explicit formula for K using $\mathbb{E}, \mathbb{F}, \mathbb{G}$ only[3]:

$$K = \frac{\det\begin{pmatrix} -\frac{1}{2}\mathbb{E}_{vv} + \mathbb{F}_{uv} - \frac{1}{2}\mathbb{G}_{uu} & \frac{1}{2}\mathbb{E}_u & \mathbb{F}_u - \frac{1}{2}\mathbb{E}_v \\ \mathbb{F}_v - \frac{1}{2}\mathbb{G}_u & \mathbb{E} & \mathbb{F} \\ \frac{1}{2}\mathbb{G}_v & \mathbb{F} & \mathbb{G} \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}\mathbb{E}_v & \frac{1}{2}\mathbb{G}_u \\ \frac{1}{2}\mathbb{E}_v & \mathbb{E} & \mathbb{F} \\ \frac{1}{2}\mathbb{G}_u & \mathbb{F} & \mathbb{G} \end{pmatrix}}{(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)^2}. \tag{20}$$

2. Surfaces of constant Gaussian curvature

2.1. Surfaces with $K = 0$

Theorem 7. *Let S be a surface covered by one single surface patch. Assume that its Gaussian curvature $K = 0$ everywhere. Then S is isometric to an open subset of a plane.*

Proof. We have seen in Lecture 13 (Proposition 12) that such S must be a ruled surface. Thus we can assume the surface patch to be $\sigma(u, v) = \alpha(u) + v\, l(u)$ with $\|l\| = \|\alpha'\| = 1$. Then we have

$$\sigma_u = \alpha'(u) + v\, l'(u), \qquad \sigma_v = l(u). \tag{21}$$

Consequently

$$\mathbb{E} = 1 + 2\, v\, \alpha' \cdot l' + v^2\, \|l'\|^2, \qquad \mathbb{F} = \alpha' \cdot l, \qquad \mathbb{G} = 1. \tag{22}$$

(20) now gives

$$\begin{aligned} 0 &= \det\begin{pmatrix} -\|l'\|^2 & \frac{1}{2}\mathbb{E}_u & \mathbb{F}_u - \frac{1}{2}\mathbb{E}_v \\ 0 & \mathbb{E} & \mathbb{F} \\ 0 & \mathbb{F} & 1 \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}\mathbb{E}_v & 0 \\ \frac{1}{2}\mathbb{E}_v & \mathbb{E} & \mathbb{F} \\ 0 & \mathbb{F} & 1 \end{pmatrix} \\ &= -\|l'\|^2\,(\mathbb{E} - \mathbb{F}^2) + \left(\frac{1}{2}\mathbb{E}_v\right)^2 \\ &= -\|l'\|^2\,(1 + 2\,v\,(\alpha' \cdot l') + v^2\,\|l'\|^2 - (\alpha' \cdot l)^2) + (\alpha' \cdot l' + v\,\|l'\|^2)^2 \\ &= -\|l'\|^2\,(1 - (\alpha' \cdot l)^2) + (\alpha' \cdot l')^2. \end{aligned} \tag{23}$$

We discuss two cases.

i. $l' = 0$. In this case σ is an open subset of a generalized cylinder and is isometric to an open subset of a plane.

ii. $l' \neq 0$. Let $\hat{l}' := \frac{l'}{\|l'\|}$, from (23) we have

$$1 - (\alpha' \cdot l)^2 - (\alpha' \cdot \hat{l}')^2 = 0. \tag{24}$$

As α' is a unit vector and so are $l, \hat{l}'$ and furthermore $l \perp \hat{l}'$, we see that $\alpha' \cdot (l \times \hat{l}') = 0$ and therefore $\alpha' \cdot (l \times l') = 0$. From §3 of Lecture 9 we see that this implies S is developable and consequently is isometric to an open subset of a plane. $\square$

3. Corollary 10.2.2 of the textbook.

2.2. Surfaces with $K > 0$ constant

THEOREM 8. *Let S be a surface covered by one single surface patch. Assume that its Gaussian curvature K is a positive constant. Then S is isometric to an open subset of a sphere.*

Proof. Clearly it suffices to consider the case $K = 1$.

Exercise 1. Rigorously justify this.

We first re-parametrize the surface so that the fundamental forms are simple.

Geodesic coordinates

PROPOSITION 9. (PROPOSITION 9.5.1 OF THE TEXTBOOK) *Let $p_0 \in S$. Then there is a neighborhood of p that can be parametrized so that the first fundamental form is $\mathrm{d}u^2 + \mathbb{G}(u,v)\,\mathrm{d}v^2$ where $\mathbb{G}(u,v)$ is a smooth function satisfying ($p_0 = \sigma(0,0)$)*

$$\mathbb{G}(0,v) = 1, \qquad \mathbb{G}_u(0,v) = 0. \tag{25}$$

Proof. Let $\gamma(v)$ be a geodesic passing p with v the arc length parameter. At each point on γ, let $\tilde{\gamma}^v(u)$ be the geodesic passing that point and is perpendicular to γ. Further assume that u is also the arc length parameter.

Exercise 2. Explain why there is a neighborhood of p that is fully covered by these $\tilde{\gamma}^v(u)$'s.

Exercise 3. Explain why each point in this neighborhood belongs to exactly one $\tilde{\gamma}^v(u)$.

Thus we obtain a surface patch $\sigma(u,v) = \tilde{\gamma}^v(u)$. As for each v, $\tilde{\gamma}^v(u)$ is arc length parametrized, there holds $\|\sigma_u\| = 1 \Longrightarrow \mathbb{E} = 1$. To show that $\mathbb{F} = 0$, we notice that by construction

$$\sigma_u(0,v) \cdot \sigma_v(0,v) = 0. \tag{26}$$

On the other hand, note that along $\tilde{\gamma}^v(u)$, we have $u' = 1, v' = 0$. The geodesic equations

$$(\mathbb{E}\,u' + \mathbb{F}\,v')_u = \frac{1}{2}(u', v')\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_u \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{27}$$

$$(\mathbb{F}\,u' + \mathbb{G}\,v')_u = \frac{1}{2}(u', v')\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_v \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{28}$$

now become

$$0 = 0 \text{ and } \mathbb{F}_u = 0. \tag{29}$$

Consequently $\mathbb{F} = 0$ for all u, v.

Exercise 4. Explain, using the intuition "geodesics are shortest paths", why $\sigma(u_0, v)$ has to be perpendicular to $\sigma(u, v_0)$.

Finally, $\sigma_v(0,v) = \frac{\mathrm{d}\gamma}{\mathrm{d}v}$ so $\mathbb{G}(0,v) = 1$. Then (27) applied to the geodesic $\gamma(v) = \sigma(0,v)$ gives $\mathbb{G}_u(0,v) = 0$. □

(20) now gives

$$-\frac{1}{2}\,\mathbb{G}_{uu}\,\mathbb{G}+\frac{1}{4}\,\mathbb{G}_u^2=\mathbb{G}^2. \tag{30}$$

Setting $\mathbb{G}=g^2$ we reach

$$g_{uu}+g=0\Longrightarrow g(u,v)=A(v)\cos u+B(v)\sin u. \tag{31}$$

Thanks to (25) there holds $g(0,v)=1, g_u(0,v)=0$ which give $A(v)=1, B(v)=0$. Consequently we have the first fundamental form to be

$$\mathrm{d}u^2+\cos^2 u\,\mathrm{d}v^2 \tag{32}$$

which is exactly the first fundamental form of $\mathbb{S}^2$ in spherical coordinates. □

2.3. Surfaces with $K<0$ constant

Theorem 10. *Let S be a surface covered by one single surface patch. Assume that its Gaussian curvature K is a negative constant. Then S is isometric to an open subset of a pseudosphere.*

The pseudosphere

Consider a surface of revolution $\sigma(u,v)=(f(u)\cos v,\, f(u)\sin v,\, g(u))$ where $f>0$ and $(f')^2+(g')^2=1$.

Exercise 5. Show that $K=-\frac{f''}{f}$.

When $K=-1$, we have $f''-f=0\Longrightarrow f(u)=a\,e^u+b\,e^{-u}$. Such a surface is called a "pseudosphere". For example when $a=1, b=0$ we have

$$g(u)=\int\sqrt{1-e^{2u}}\,\mathrm{d}u=\sqrt{1-e^{2u}}-\ln\left(e^{-u}+\sqrt{e^{-2u}-1}\right). \tag{33}$$

Exercise 6. Draw a sketch of the pseudosphere in this case.

Proof. (of Theorem 10) Same idea as the proof of Theorem 8. See p.258 of the textbook. □

2.4. Compact surfaces

Theorem 11. [4]*Every connected compact surface whose Gaussian curvature is constant is a sphere.*

Exercise 7. Every compact surface has a point where $K\geqslant 0$. (Hint: Consider $p\in S$ that is furthest away from the origin)

Proof. Consider the function $J=(\kappa_1-\kappa_2)^2$. Let p be where J reaches maximum–this is possible as J is continuous. If this maximum is 0 then we have $\kappa_1=\kappa_2$ and S must be part of a sphere. If not, thanks to $\kappa_1\kappa_2=K$ is constant, there must hold that κ_1 is at local maximum and κ_2 at local minimum at p. We reach contradiction using the following lemma.

4. Theorem 10.3.4 of the textbook.

LEMMA 12. [5]*Let $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch containing a point p that is not an umbilic*[6]. *Let $\kappa_1 \geqslant \kappa_2$ be the principal curvatures of σ and suppose that κ_1 has a local maximum at p and κ_2 has a local minimum at p. Then $K(p) \leqslant 0$.* □

Proof. (OF LEMMA 12) Taking a small neighborhood of p such that $\kappa_1 > \kappa_2$ in it. Take the coordinate system along the principal vectors so that the two fundamental forms are

$$\mathbb{E}\, \mathrm{d}u^2 + \mathbb{G}\, \mathrm{d}v^2, \qquad \mathbb{L}\, \mathrm{d}u^2 + \mathbb{N}\, \mathrm{d}v^2. \tag{34}$$

Thus we have $\kappa_1 = \frac{\mathbb{L}}{\mathbb{E}}$ and $\kappa_2 = \frac{\mathbb{N}}{\mathbb{G}}$.

Exercise 8. What about $\kappa_1 > \kappa_2$?

Taking derivatives of the Codazzi-Mainradi equations we obtain

$$\mathbb{L}_v = \frac{1}{2}\, \mathbb{E}_v \left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}} \right), \qquad \mathbb{N}_u = \frac{1}{2}\, \mathbb{G}_u \left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}} \right) \tag{35}$$

and consequently

$$\mathbb{E}_v = \left(\frac{2\, \mathbb{E}}{\kappa_2 - \kappa_1} \right) (\kappa_1)_v, \qquad \mathbb{G}_u = \left(\frac{2\, \mathbb{G}}{\kappa_1 - \kappa_2} \right) (\kappa_2)_u. \tag{36}$$

At p we have $(\kappa_1)_v = (\kappa_2)_u = 0 \Longrightarrow \mathbb{E}_v = \mathbb{G}_u = 0$ which leads to

$$\begin{aligned} K(p) &= -\frac{1}{2\sqrt{\mathbb{E}\,\mathbb{G}}} \left(\frac{\partial}{\partial u} \left(\frac{\mathbb{G}_u}{\sqrt{\mathbb{E}\,\mathbb{G}}} \right) + \frac{\partial}{\partial v} \left(\frac{\mathbb{E}_v}{\sqrt{\mathbb{E}\,\mathbb{G}}} \right) \right) \\ &= -\frac{\mathbb{G}_{uu} + \mathbb{E}_{vv}}{2\, \mathbb{E}\, \mathbb{G}} \\ &= -\frac{\mathbb{G}\, (\kappa_2)_{uu} - \mathbb{E}\, (\kappa_1)_{vv}}{\mathbb{E}\, \mathbb{G}\, (\kappa_1 - \kappa_2)} \leqslant 0, \end{aligned} \tag{37}$$

where the last inequality comes from κ_1 taking a local maximum and κ_2 taking a local minimum at p . □

Remark 13. Similar to the proof of Theorem 11 we can show that a compact surface with $K > 0$ everywhere and constant H is a sphere.

5. Lemma 10.3.5 of the textbook.

6. that is $\kappa_1 \neq \kappa_2$.