Lectures 12–13: Curvatures for Surfaces

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce several quantities that characterize the curving of a surface patch.

The required textbook sections are §8.1–8.2. The optional sections are §8.3–8.6.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Gaussian and mean curvatures

- One can show that the first and second fundamental forms completely determines the surface.
- However these are complicated quantities. It turns out that there are more compact ways to understand the curving of surfaces.
- Mean curvature. Consider the normal curvatures κ_n at one point. Pick an arbitrary direction $w_0 \in T_p S$ and let θ be the counterclockwise angle from w_0 to the tangent direction w along which κ_n is calculated. Then we have $\kappa_n = \kappa_n(\theta)$. We will define the mean curvature as the average of all the κ_n's:

$$H := \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, \mathrm{d}\theta. \tag{1}$$

Remark 1. It is important to realize that H is independent of the choice of w_0. That is, if we take another $w_1 \in T_p S$ and let θ_1 be the angle from w_1 to w, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta_1)\, \mathrm{d}\theta_1 = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, \mathrm{d}\theta = H. \tag{2}$$

Exercise 1. Prove this.

- Gaussian curvature. Consider the Gauss map $\mathcal{G}: S \mapsto \mathbb{S}^2$ and the corresponding Weingarten map $\mathcal{W}$. Recall that

$$\mathcal{W}(\sigma_u) = -N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, \qquad \mathcal{W}(\sigma_v) = -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v, \tag{3}$$

where $a_{11}, \ldots, a_{22}$ can be calculated through

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}. \tag{4}$$

Now let U be a region in the u-v plane. Then $N: U \mapsto \mathbb{S}^2$ is a surface patch for $\mathbb{S}^2$. We calculate

$$N_u \times N_v = (a_{11}\,a_{22} - a_{21}\,a_{12})\,\sigma_u \times \sigma_v. \tag{5}$$

Therefore

$$\|N_u \times N_v\| = |a_{11}\,a_{22} - a_{21}\,a_{12}|\,\|\sigma_u \times \sigma_v\|. \tag{6}$$

Consequently

$$\text{Area of } N(U) = \int_U |a_{11}\,a_{22} - a_{21}\,a_{12}|\,\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v \tag{7}$$

and if we take U_r to be a small disc $D_p((u_0, v_0))$ centering at (u_0, v_0) with radius r, we would have

$$\lim_{r \longrightarrow 0} \frac{\text{Area of } N(U)}{\text{Area of } \sigma(U)} = |a_{11}\,a_{22} - a_{21}\,a_{12}|. \tag{8}$$

Exercise 2. Prove this.

We will call the number

$$K := a_{11}\, a_{22} - a_{21}\, a_{12} \tag{9}$$

the Gaussian curvature of S at p_0.

2. Principal curvatures

- We try to understand the mean curvature H. To do this we need a formula for $\kappa_n(\theta)$.
- Recall that if we take $\|w(\theta)\| = 1$,

$$\kappa_n(\theta) = \frac{\langle\langle w(\theta), w(\theta)\rangle\rangle}{\langle w(\theta), w(\theta)\rangle} = \langle\langle w(\theta), w(\theta)\rangle\rangle. \tag{10}$$

- Now let e_1, e_2 be an orthonormal basis for the tangent plane $T_p S$, we can set $w(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$. Substituting into (10) we have

$$\kappa_n(\theta) = \langle\langle e_1, e_1\rangle\rangle \cos^2\theta + 2\, \langle\langle e_1, e_2\rangle\rangle \cos\theta \sin\theta + \langle\langle e_2, e_2\rangle\rangle \sin^2\theta. \tag{11}$$

Integrating we get

$$H = \frac{1}{2\,\pi} \int_0^{2\pi} \kappa_n(\theta)\, \mathrm{d}\theta = \frac{1}{2}\, [\langle\langle e_1, e_1\rangle\rangle + \langle\langle e_2, e_2\rangle\rangle\,]. \tag{12}$$

- Taking derivative

$$\kappa_n'(\theta) = (\langle\langle e_2, e_2\rangle\rangle - \langle\langle e_1, e_1\rangle\rangle) \cos 2\,\theta + 2\, \langle\langle e_1, e_2\rangle\rangle \sin 2\,\theta. \tag{13}$$

We see that $\kappa_n'(\theta) = 0$ has four solutions in $[0, 2\,\pi]$: $\theta_0, \theta_0 + \pi/2, \theta_0 + \pi, \theta_0 + 3\,\pi/2$. As clearly $\kappa_n(\theta + \pi) = \kappa_n(\theta)$, and $\kappa_n(\theta)$ must achieve both maximum and minimum, there are θ_1, θ_2 such that $\theta_2 = \theta_1 + \pi/2$ and $\kappa_1 = \kappa(\theta_1) = \max \kappa(\theta)$, $\kappa_2 = \kappa(\theta_2) = \min \kappa(\theta)$. Now we can take $\tilde{e}_1 := w(\theta_1)$ and $\tilde{e}_2 := w(\theta_2)$ and re-do the calculation above using $\tilde{e}_1$, $\tilde{e}_2$ as the orthonomal basis and conclude that

$$H = \frac{\kappa_1 + \kappa_2}{2}. \tag{14}$$

We call κ_1, κ_2 the *principal curvatures*, and the corresponding directions $t_1 := w(\theta_1)$, $t_2 := w(\theta_2)$ the *principal vectors* corresponding to κ_1 and κ_2.

3. How to calculate $H, K, \kappa_1, \kappa_2, t_1, t_2$.

- The calculation of Gaussian curvature is easy. Recall that

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}. \tag{15}$$

We easily obtain

$$K = \det \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{\det \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}}{\det \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}} = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}. \tag{16}$$

- For the principal and mean curvatures, we try to calculate κ_1, κ_2 in a different way. Let $w := a\,\sigma_u + b\,\sigma_v$. We try to find the maximum and minimum of

$$\kappa(w) = \mathbb{L}\,a^2 + 2\,\mathbb{M}\,a\,b + \mathbb{N}\,b^2 \tag{17}$$

under the constraint $\|w\| = 1$, that is $\mathbb{E}\,a^2 + 2\,\mathbb{F}\,a\,b + \mathbb{G}\,b^2 = 1$. To do this we apply the method of Lagrange multiplier:

$$L(a,b) := [\mathbb{L}\,a^2 + 2\,\mathbb{M}\,a\,b + \mathbb{N}\,b^2] - \lambda\,[\mathbb{E}\,a^2 + 2\,\mathbb{F}\,a\,b + \mathbb{G}\,b^2]. \tag{18}$$

Thus

$$\frac{\partial L}{\partial a} = 2\,[\mathbb{L}\,a + \mathbb{M}\,b - \lambda(\mathbb{E}\,a + \mathbb{F}\,b)], \tag{19}$$

$$\frac{\partial L}{\partial b} = 2\,[\mathbb{M}\,a + \mathbb{N}\,b - \lambda\,(\mathbb{F}\,a + \mathbb{G}\,b)]. \tag{20}$$

Setting them to zero we see that λ and $\begin{pmatrix} a \\ b \end{pmatrix}$ solves

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \lambda \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{21}$$

This means λ solves

$$\det \begin{pmatrix} \mathbb{L} - \lambda\,\mathbb{E} & \mathbb{M} - \lambda\,\mathbb{F} \\ \mathbb{M} - \lambda\,\mathbb{F} & \mathbb{N} - \lambda\,\mathbb{G} \end{pmatrix} = 0 \tag{22}$$

which simplifies to the quadratic equation

$$(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)\,\lambda^2 - (\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F})\,\lambda + (\mathbb{L}\,\mathbb{N} - \mathbb{M}^2) = 0. \tag{23}$$

- What is λ?

If we set $\lambda = \kappa_1$, we see that $L(a,b) \leqslant 0$ and $L(a_1, b_1) = 0$ for some $\|a_1\,\sigma_u + b_1\,\sigma_v\| = 1$. Thus (a_1, b_1) maximizes $L(a,b)$ on the curve $\|a_1\,\sigma_u + b_1\,\sigma_v\| = 1$.[1] Now notice that for every $c \geqslant 0$ there holds $L(c\,a, c\,b) = c^2\,L(a,b)$. Consequently (a_1, b_1) maximizes $L(a,b)$ over the whole $\mathbb{R}^2$. Thus there must hold $\frac{\partial L}{\partial a}(a_1, b_1) = \frac{\partial L}{\partial b}(a_1, b_1) = 0$ and in particular, κ_1 solves (23).

Similarly we can show that κ_2 solves (23) too. But (23) has at most two real solutions. So κ_1, κ_2 are exactly the solutions of (23).

The principal vectors are now given by

$$t_1 = a_1\,\sigma_u + b_1\,\sigma_v, \qquad t_2 = a_2\,\sigma_u + b_2\,\sigma_v \tag{24}$$

where a_i, b_i solves

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0, \qquad i = 1, 2. \tag{25}$$

- Summarizing, we see that

$$H = \frac{\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}. \tag{26}$$

1. Try to prove (or convince yourself) that this curve is an ellipsis.

- An interesting consequence of the above calculation is that $K=\kappa_1\,\kappa_2$.
- Alternative characterization of κ_1, κ_2.

  If we set $\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix}$, we have

$$\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = 0. \tag{27}$$

  Therefore $\kappa_{1,2}$ are eigenvalues of the matrix $\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}$. Recall that the geometrical meanings of a_{ij} are given through

$$-N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, \qquad -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v. \tag{28}$$

  Thus we have

$$H = \mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right], \qquad K = \det\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{29}$$

Principal curvature, mean curvature, Gaussian curvature

- Principal curvatures.

$$\det\begin{pmatrix} \mathbb{L} - \kappa_i\,\mathbb{E} & \mathbb{M} - \kappa_i\,\mathbb{F} \\ \mathbb{M} - \kappa_i\,\mathbb{F} & \mathbb{N} - \kappa_i\,\mathbb{G} \end{pmatrix} = 0, \tag{30}$$

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right]\begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0, \tag{31}$$

$$t_i = a_i\,\sigma_u + b_i\,\sigma_v. \tag{32}$$

- Mean curvature.

$$H = \frac{1}{2\,\pi}\int_0^{2\pi} \kappa_n(\theta)\,\mathrm{d}\theta = \frac{\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{33}$$

- Gaussian curvature.

$$K = \lim_{r \longrightarrow 0} \frac{\text{Area of } N(B_r)}{\text{Area of } \sigma(B_r)} = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = \det\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{34}$$

- Relations.

$$H = \frac{\kappa_1 + \kappa_2}{2}, \qquad K = \kappa_1\,\kappa_2, \qquad \kappa_{1,2} = \frac{H \pm \sqrt{H^2 - 4\,K}}{2}. \tag{35}$$

$$\kappa_n((\cos\theta)\,t_1 + (\sin\theta)\,t_2) = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta. \tag{36}$$

Remark 2. We have seen last time that if $\kappa_1 = \kappa_2$ everywhere, then S is part of plane or sphere.

4. Examples

Example 3. Let $\sigma(u, v) = (u, v, f(u, v))$ be the graph of some smooth function $f(x, y)$: $U \mapsto \mathbb{R}$. Then

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \qquad H = \frac{(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy}}{2 (1 + f_x^2 + f_y^2)^{3/2}}. \tag{37}$$

Proof. We calculate

$$\sigma_u = (1, 0, f_x), \qquad \sigma_v = (0, 1, f_y), \qquad N = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}, \tag{38}$$

$$\sigma_{uu} = (0, 0, f_{xx}), \qquad \sigma_{uv} = (0, 0, f_{xy}), \qquad \sigma_{vv} = (0, 0, f_{yy}). \tag{39}$$

Therefore

$$\mathbb{E} = 1 + f_x^2, \qquad \mathbb{F} = f_x f_y, \qquad \mathbb{G} = 1 + f_y^2, \tag{40}$$

$$\mathbb{L} = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \qquad \mathbb{M} = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \qquad \mathbb{N} = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}. \tag{41}$$

Consequently

$$K = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} \tag{42}$$

and

$$H = \frac{\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy}}{2 (1 + f_x^2 + f_y^2)^{3/2}}, \tag{43}$$

as desired. □

Example 4. Consider the surface $z = \alpha x^2 + \beta y^2$ where $\alpha, \beta \in \mathbb{R}$. Calculate $H, K, \kappa_1, \kappa_2, t_1$, t_2 at the origin.

Solution. We take the surface patch $\sigma(u, v) = (u, v, \alpha u^2 + \beta v^2)$. Then we have

$$\sigma_u = (1, 0, 2\,\alpha\,u), \qquad \sigma_v = (0, 1, 2\,\beta\,v), \qquad N = \frac{(-2\,\alpha\,u, -2\,\beta\,v, 1)}{\sqrt{1 + 4\,\alpha^2 u^2 + 4\,\beta^2 v^2}}, \tag{44}$$

$$\sigma_{uu} = (0, 0, 2\,\alpha), \qquad \sigma_{uv} = (0, 0, 0), \qquad \sigma_{vv} = (0, 0, 2\,\beta). \tag{45}$$

Thus at the origin which corresponds to $u = v = 0$, we have

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = 1, \tag{46}$$

$$\mathbb{L} = 2\,\alpha, \qquad \mathbb{M} = 0, \qquad \mathbb{N} = 2\,\beta. \tag{47}$$

Consequently we have (wlog assume $\alpha > \beta$),

$$\kappa_1 = 2\,\alpha, \qquad t_1 = (1, 0, 0); \qquad \kappa_2 = 2\,\beta, \qquad t_2 = (0, 1, 0), \tag{48}$$

$$H = \alpha + \beta, \qquad K = 4\,\alpha\,\beta. \tag{49}$$

Example 5. [2]Let S be an oriented surface and let $\lambda \in \mathbb{R}$. The parallel surface S^λ of S is

$$S^\lambda = \{p + \lambda\, N_p |\, p \in S\} \tag{50}$$

where N_p is the unit normal of S at the point p. Then[3]

$$K^\lambda = \frac{K}{1 - 2\,\lambda\, H + \lambda^2\, K}, \qquad H^\lambda = \frac{H - \lambda\, K}{1 - 2\,\lambda\, H + \lambda^2\, K}. \tag{51}$$

Here we assume $|\lambda|$ to be small enough such that $1 - 2\, H\, \lambda + \lambda^2\, K > 0$.

Proof. We take the surface patch $\sigma^\lambda(u, v) = \sigma(u, v) + \lambda\, N(u, v)$. Then

$$\sigma_u^\lambda = \sigma_u + \lambda\, N_u, \qquad \sigma_v^\lambda = \sigma_v + \lambda\, N_v. \tag{52}$$

Now recall that $-N_u = a_{11}\, \sigma_u + a_{12}\, \sigma_v$, $-N_v = a_{21}\, \sigma_u + a_{22}\, \sigma_v$, we obtain

$$\sigma_u^\lambda \times \sigma_v^\lambda = [(1 - \lambda\, a_{11})\, (1 - \lambda\, a_{22}) - \lambda^2\, a_{12}\, a_{21}]\, (\sigma_u \times \sigma_v) = [1 - 2\, H\, \lambda + \lambda^2\, K]\, (\sigma_u \times \sigma_v). \tag{53}$$

Thus when λ is small, there holds $N^\lambda = N$.

Consequently we have, using $N_u = N_u^\lambda$ and $N_v = N_v^\lambda$,

$$(1 - \lambda\, a_{11})\, (-N_u^\lambda) - \lambda\, a_{12}\, (-N_v^\lambda) = a_{11}\, \sigma_u^\lambda + a_{12}\, \sigma_v^\lambda, \tag{54}$$

$$-\lambda a_{21}(-N_u^\lambda) + (1 - \lambda\, a_{11})(-N_v^\lambda) = a_{21}\, \sigma_u^\lambda + a_{22}\, \sigma_v^\lambda. \tag{55}$$

From these we have

$$-N_u^\lambda = a_{11}^\lambda\, \sigma_u^\lambda + a_{12}^\lambda\, \sigma_v^\lambda, \qquad -N_v^\lambda = a_{21}^\lambda\, \sigma_u^\lambda + a_{22}^\lambda\, \sigma_v^\lambda, \tag{56}$$

where

$$\begin{pmatrix} a_{11}^\lambda & a_{12}^\lambda \\ a_{21}^\lambda & a_{22}^\lambda \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right]^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{57}$$

Now let κ_1, κ_2 be eigenvalues of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with eigenvectors $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$, we have

$$\begin{aligned} \begin{pmatrix} a_{11}^\lambda & a_{12}^\lambda \\ a_{21}^\lambda & a_{22}^\lambda \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right]^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \\ &= \kappa_i \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right]^{-1} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \\ &= \kappa_i\, (1 - \lambda\, \kappa_i)^{-1} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}. \end{aligned} \tag{58}$$

Consequently the eigenvalues for $\begin{pmatrix} a_{11}^\lambda & a_{12}^\lambda \\ a_{21}^\lambda & a_{22}^\lambda \end{pmatrix}$ are given by $\kappa_i^\lambda = \frac{\kappa_i}{1 - \lambda\, \kappa_i}$. Thus finally we have

$$H^\lambda = \frac{1}{2} \left[\frac{\kappa_1}{1 - \lambda\, \kappa_1} + \frac{\kappa_2}{1 - \lambda\, \kappa_2} \right] = \frac{H - \lambda\, K}{1 - 2\, H\, \lambda + K\, \lambda^2}, \tag{59}$$

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2. Definition 8.5.1 of the textbook

3. Proposition 8.5.2 of the textbook.

and

$$K^{\lambda} = \frac{\kappa_1 \kappa_2}{(1 - \lambda \kappa_1)(1 - \lambda \kappa_2)} = \frac{K}{1 - 2 \lambda H + \lambda^2 K}, \tag{60}$$

as desired. □

Exercise 3. Solve the problem when $1 - 2 H \lambda + \lambda^2 K < 0$.

5. Minimal surfaces (optional)

5.1. The problem

- The so-called "Plateau's problem" asks the following questions: Given a closed curve in the space $\mathbb{R}^3$, among the infinitely many surfaces having this curve as its boundary, which one has the minimal area?

Example 6. Let C be a simple closed plane curve. Then the minimal surface with C as its boundary is the part of the plane enclosed by C.

Proof. Let U be the region of the plane that is enclosed by C. Let $\sigma: U \mapsto \mathbb{R}^3, \sigma(u, v) = (u, v, f(u, v))$ be an arbitrary surface patch. All we need to show is that the area of $\sigma(u, v)$ is no less than the area of U.

Exercise 4. Point out as many gaps as you can in the above set up. Can you fill them?

Now we calculate

$$\sigma_u = (1, 0, f_x), \qquad \sigma_v = (0, 1, f_y) \tag{61}$$

and

$$\sigma_u \times \sigma_v = (-f_x, -f_y, 1). \tag{62}$$

Therefore we have

$$\begin{aligned} \text{Area of } \sigma &= \int_U \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_U \sqrt{1 + f_x^2 + f_y^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_U \mathrm{d}u \, \mathrm{d}v = \text{Area of } U. \end{aligned} \tag{63}$$

Thus ends the proof. □

Exercise 5. What if C is a curve on the cylinder? the sphere?

5.2. Variational analysis

- When the curve is not a plane curve the situation becomes much more complicated.
- We rely one variational analysis to obtain some characterizing equation for this minimal surface.
- Variational analysis is an upgrade of "taking derivative and set it to zero" in first year calculus.

- Let $\sigma^0(u, v)$: $U \mapsto \mathbb{R}^3$ be a surface patch for the minimal surface. Thus we have $\sigma^0(\partial U) = C$. Now let $\sigma(u, v)$: $U \mapsto \mathbb{R}^3$ be an arbitrary surface patch satisfying $\sigma(\partial U) = \{0\}$. Thus at least for $\tau \in \mathbb{R}$ with $|\tau|$ small, we have $\sigma^\tau := \sigma^0 + \tau\,\sigma$ to be another surface patch with the same boundary C.
- Now define

$$\mathcal{A}(\tau) := \int_U \|\sigma_u^\tau \times \sigma_v^\tau\| \,\mathrm{d}u\,\mathrm{d}v \tag{64}$$

we clearly have $\mathcal{A}(0) \leqslant \mathcal{A}(\tau)$. Consequently we must have $\mathcal{A}'(\tau) = 0$.

- We calculate

$$\sigma_u^\tau = \sigma_u^0 + \tau\,\sigma_u, \qquad \sigma_v^\tau = \sigma_v^0 + \tau\,\sigma_v \tag{65}$$

and therefore

$$\sigma_u^\tau \times \sigma_v^\tau = \sigma_u^0 \times \sigma_v^0 + \tau\,[\sigma_u^0 \times \sigma_v + \sigma_u \times \sigma_v^0] + \tau^2\,\sigma_u \times \sigma_v. \tag{66}$$

Let's denote for now

$$V_0 := \sigma_u^0 \times \sigma_v^0, \qquad V_1 := \sigma_u^0 \times \sigma_v + \sigma_u \times \sigma_v^0, \qquad V_2 := \sigma_u \times \sigma_v. \tag{67}$$

- Thus we have

$$\mathcal{A}(\tau) = \int_U \sqrt{V_0 \cdot V_0 + 2\,\tau V_0 \cdot V_1 + O(\tau^2)}\,\mathrm{d}u\,\mathrm{d}v \tag{68}$$

Taking τ-derivative we obtain

$$\mathcal{A}'(0) = \int_U \frac{V_0 \cdot V_1}{\sqrt{V_0 \cdot V_0}}\,\mathrm{d}u\,\mathrm{d}v = \int_U \frac{V_0 \cdot V_1}{\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}}\,\mathrm{d}u\,\mathrm{d}v. \tag{69}$$

- To calculate $V_0 \cdot V_1$ we use the vector identity

$$(a \times b) \cdot (c \times d) = (a \cdot c)\,(b \cdot d) - (a \cdot d)\,(b \cdot c). \tag{70}$$

This leads to

$$\begin{aligned} V_0 \cdot V_1 &= (\sigma_u^0 \times \sigma_v^0) \cdot (\sigma_u^0 \times \sigma_v) + (\sigma_u^0 \times \sigma_v^0) \cdot (\sigma_u \times \sigma_v^0) \\ &= (\sigma_u^0 \cdot \sigma_u^0)\,(\sigma_v^0 \cdot \sigma_v) - (\sigma_u^0 \cdot \sigma_v)\,(\sigma_v^0 \cdot \sigma_u^0) + (\sigma_u^0 \cdot \sigma_u)\,(\sigma_v^0 \cdot \sigma_v^0) - (\sigma_u^0 \cdot \sigma_v^0)\,(\sigma_u \cdot \sigma_v^0) \\ &= \mathbb{E}\,(\sigma_v^0 \cdot \sigma_v) - \mathbb{F}\,(\sigma_u^0 \cdot \sigma_v + \sigma_u \cdot \sigma_v^0) + \mathbb{G}\,(\sigma_u^0 \cdot \sigma_u). \end{aligned} \tag{71}$$

- To simplify (71) we notice that

$$\begin{pmatrix} \mathbb{G} & -\mathbb{F} \\ -\mathbb{F} & \mathbb{E} \end{pmatrix} = (\mathbb{E}\,\mathbb{G} - \mathbb{F}^2) \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}. \tag{72}$$

This inspires us to write

$$\sigma_u = a_{11}\,\sigma_u^0 + a_{12}\,\sigma_v^0 + a_{13}\,N^0, \qquad \sigma_v = a_{21}\,\sigma_u^0 + a_{22}\,\sigma_v^0 + a_{23}\,N^0. \tag{73}$$

Therefore

$$\sigma_u \cdot \sigma_u^0 = \mathbb{E}\,a_{11} + \mathbb{F}\,a_{12}, \qquad \sigma_u \cdot \sigma_v^0 = \mathbb{F}\,a_{11} + \mathbb{G}\,a_{12}, \tag{74}$$

$$\sigma_v \cdot \sigma_u^0 = \mathbb{E}\,a_{21} + \mathbb{F}\,a_{22}, \qquad \sigma_v \cdot \sigma_v^0 = \mathbb{F}\,a_{21} + \mathbb{G}\,a_{22}. \tag{75}$$

Substituting into (71) we have

$$V_0 \cdot V_1 = (a_{11} + a_{22})\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2). \tag{76}$$

Thus

$$\mathcal{A}'(0) = \int_U (a_{11} + a_{22})\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v \tag{77}$$

- Finally, we notice that as σ is arbitrary, we could restrict ourselves to $\sigma(u, v) = f(u, v)\,N^0(u, v)$ where $f(u, v)$ is a scalar function vanishing on ∂U. Thus we have

  $$\sigma_u = f\,N_u^0 + f_u\,N^0, \qquad \sigma_v = f\,N_v^0 + f_v\,N^0. \tag{78}$$

  Comparing with (73), we see that $a_{11} + a_{22} = -2\,f\,H$. Consequently

  $$\mathcal{A}'(0) = -2\int_U f\,H\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v = -2\int_{S^0} f\,H\,\mathrm{d}S \tag{79}$$

  where the last is the surface integral as defined in multivariable calculus.

- Since f is arbitrary, for $\mathcal{A}'(0) = 0$ we must have $H = 0$.

Definition 7. (Minimal surface) [4]*A **minimal surface** is a surface whose mean curvature is zero everywhere.*

5.3. Examples

Example 8. A plane region is a minimal surface; The cylinder is not a minimal surface.

Example 9. [5]Any ruled minimal surface is an open subset of a plane or a helicoid.

Proof. Let $\sigma(u, v) = \alpha(u) + v\,l(u)$ be a surface patch for the ruled minimal surface. We calculate

$$\begin{array}{lll} \sigma_u = \alpha' + v\,l', & \sigma_v = l, & \sigma_u \times \sigma_v = (\alpha' + v\,l') \times l \\ \sigma_{uu} = \alpha'' + v\,l'', & \sigma_{uv} = l', & \sigma_{vv} = 0 \end{array}. \tag{80}$$

Therefore

$$\begin{array}{lll} \mathbb{E} = (\alpha' + v\,l') \cdot (\alpha' + v\,l'), & \mathbb{F} = (\alpha' + v\,l') \cdot l, & \mathbb{G} = l \cdot l, \\ \mathbb{L} = \dfrac{(\alpha'' + v\,l'') \cdot [(\alpha' + v\,l') \times l]}{\|(\alpha' + v\,l') \times l\|}, & \mathbb{M} = \dfrac{l' \cdot [(\alpha' + v\,l') \times l]}{\|(\alpha' + v\,l') \times l\|}, & \mathbb{N} = 0. \end{array} \tag{81}$$

Now we make simplifying assumptions.

- It is clear that we can assume $\|l(u)\| = 1$. This simplifies (81) to

  $$\begin{array}{lll} \mathbb{E} = (\alpha' + v\,l') \cdot (\alpha' + v\,l'), & \mathbb{F} = \alpha' \cdot l, & \mathbb{G} = 1, \\ \mathbb{L} = \dfrac{(\alpha'' + v\,l'') \cdot [(\alpha' + v\,l') \times l]}{\|(\alpha' + v\,l') \times l\|}, & \mathbb{M} = \dfrac{l' \cdot [(\alpha' + v\,l') \times l]}{\|(\alpha' + v\,l') \times l\|}, & \mathbb{N} = 0. \end{array} \tag{82}$$

4. Definition 12.1.2 in the textbook.

5. Proposition 12.2.4 in the textbook.

- We can further assume $\|l'(u)\| = 1$.

Now $H = 0$ implies $\mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F} = 0$ which becomes

$$[\alpha'' + v\,l'' - 2\,(\alpha' \cdot l)\,l'] \cdot [(\alpha' + v\,l') \times l] = 0. \tag{83}$$

Expanding (83) into powers of v, we see that

$$[(l' \times l) \cdot l'']\,v^2 + [(l' \times l) \cdot \alpha'' + (\alpha' \times l) \cdot l'']\,v + [(\alpha' \times l) \cdot \alpha'' - 2\,(\alpha' \cdot l)\,((\alpha' \times l) \cdot l')] = 0. \tag{84}$$

(84) must hold for all v. Consequently

$$(l' \times l) \cdot l'' = 0, \tag{85}$$
$$(l' \times l) \cdot \alpha'' + (\alpha' \times l) \cdot l'' = 0, \tag{86}$$
$$(\alpha' \times l) \cdot \alpha'' - 2\,(\alpha' \cdot l)\,((\alpha' \times l) \cdot l') = 0. \tag{87}$$

Now by (85) we conclude that $l(u)$ has zero torsion and is a plane curve. But by our assumption $l(u)$ is also a spherical curve. Consequently $l(u)$ is a circle. In fact since l belongs to the same plane as this circle, $l(u)$ must be a big circle on the unit sphere. Consequently we have $l'' = -l$.

Now notice that $\{l,\ l',\ N = l \times l'\}$ form an orthonormal basis. Thus we write $\alpha' = \lambda\,l + \mu\,l' + \gamma\,N$. Taking derivative and using the facts that N is a constant vector as well as $l'' = -l$, we have

$$\alpha'' = (\lambda' - \mu)\,l + (\lambda + \mu')\,l' + \gamma'\,N. \tag{88}$$

By (86) we have $(l' \times l) \cdot \alpha'' = 0$ which means $\gamma' = 0$ so $\gamma = \gamma_0$ is a constant.

Finally we take span$\{l, l'\}$ to be the x-y plane. Thus we have

$$\alpha(u) = (f(u), g(u), \gamma_0\,u + \gamma_1) \tag{89}$$

where γ_0, γ_1 are constants, and $l(u) = (\cos u, \sin u, 0)$. Now there are two cases.

- $\gamma_0 = 0$. Clearly σ is part of a plane (recall that l also is in the x-y plane;
- $\gamma_0 \neq 0$. In this case (87) simplifies to

$$g''\cos u - f''\sin u = 2\,(f'\cos u + g'\sin u). \tag{90}$$

  Now notice that we can always pick $\alpha(u)$ such that $\alpha'(u) \cdot l(u) = 0$. This gives $f'\cos u + g'\sin u = 0$ and consequently

$$(g'\cos u - f'\sin u)' = 0 \Longrightarrow g'\cos u - f'\sin u = c_0 \tag{91}$$

  Putting together

$$f'\cos u + g'\sin u = 0 \tag{92}$$
$$-f'\sin u + g'\cos u = c_0 \tag{93}$$

  we reach

$$f' = -c_0\sin u, \qquad g' = c_0\cos u \tag{94}$$

  which means

$$f = c_1 + c_0\cos u, \qquad g = c_2 + c_0\sin u. \tag{95}$$

So finally we have

$$\sigma(u,v) = (c_1 + (v + c_0)\cos u, c_2 + (v + c_0)\sin u, \gamma_0 u + \gamma_1) \tag{96}$$

which is the same as

$$\sigma(u,v) = (c_1 + v\,\mathrm{cosu}, c_2 + v\,\mathrm{sinu}, \gamma_1 + \gamma_0\,u), \tag{97}$$

a helicoid. □

Remark 10. Note that the helicoid is not developable.

6. Developable surfaces (optional)

Recall that we have proved that the only developable surfaces are the plane, the (generalized) cylinder, the (generalized) cone, and a class of surfaces called "tangent developables". In the proof we left one big gap: the claim that any developable surface must be ruled. Now we finally are able to fill this gap.

In the following we assume S is a developable surface, that is a surface having local isometries with the flat plane. Recall that a local isometry $f: S_1 \mapsto S_2$ is characterized by the fact that for every surface patch σ_1 for S_1, if we denote by $\sigma_2 := f \circ \sigma_1$, then the first fundamental forms are identical: $\mathbb{E}_1 = \mathbb{E}_2, \mathbb{F}_1 = \mathbb{F}_2, \mathbb{G}_1 = \mathbb{G}_2$.

LEMMA 11. *S must have Gaussian curvature zero everywhere.*

Proof. Left as exercise. □

PROPOSITION 12. (PROPOSITION 8.4.2 OF THE TEXTBOOK) *Let $p \in S$ be such that the principal curvatures $\kappa_1 \neq \kappa_2$ there. Then there is a straight line segment passing p while at the same time contained in S. In other words, S is a ruled surface.*

Proof.

i. Pick $\sigma(u,v)$ such that the first and second fundamental forms are

$$\mathbb{E}\,\mathrm{d}u^2 + \mathbb{G}\,\mathrm{d}v^2, \qquad \mathbb{L}\,\mathrm{d}u^2 + \mathbb{N}\,\mathrm{d}v^2. \tag{98}$$

Exercise 6. Why can this be done?

ii. Since $K = 0$, there must hold $\mathbb{L}\,\mathbb{N} = 0$. Note that if both $\mathbb{L}, \mathbb{N} = 0$, then $\kappa_1 = \kappa_2 = 0$. Therefore we can assume $\mathbb{L} \neq 0$ or $\mathbb{N} \neq 0$. We study the case $\mathbb{L} \neq 0$ and leave the case $\mathbb{N} \neq 0$ as exercise. Note that if $\mathbb{L} \neq 0$ then necessarily $\mathbb{N} = 0$.

iii. The second fundamental form is now $\mathbb{L}\,\mathrm{d}u^2$. We will prove that $\sigma(u_0, v)$ is a straight line. Since

$$N_u = -\mathbb{E}^{-1}\,\mathbb{L}\,\sigma_u, \qquad N_v = 0, \tag{99}$$

we have $T = \frac{\sigma_v}{\mathbb{G}^{1/2}}$ and $T_v \cdot N_u = 0, T_v \cdot N = 0, T_v \cdot T = 0$. This implies $T_v = 0$ and we are done. □

Exercise 7. What happens if $\kappa_1 = \kappa_2 = 0$?