Lecture 7: Differential Geometry of Curves II

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study how a curve curves. For simplicity we assume the curve is already in arc length parameter. We will show that the curving of a general curve can be characterized by two numbers, the curvature and the torsion.

The required textbook sections are §2.3. The optional sections are §2.2.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Torsion

Torsion measures how quickly a curve "twists".

1.1. The osculating plane

- MOTIVATION. Consider a point on a space curve. We have seen that to measure how quickly it curves, we should measure the rate of change for the unit tangent vector. Similarly, to measure how quickly it "twists", we should measure the change rate of the "tangent plane".
- THE OSCULATING PLANE.
  - Let $x(s)$ be a space curve. Its osculating plane at $x(s_0)$ is the plane passing $x(s_0)$ that is spanned by the unit tangent vector $T(s_0) := x'(s_0)$ and the unit normal vector $N(s_0) := \frac{x''(s_0)}{\|x''(s_0)\|}$.
  - We see that the osculating plane contains the tangent line.
  - The unit normal vector of the osculating plane is then given by

    $$B(s) := T(s) \times N(s) \tag{1}$$

    which we call the *unit binormal vector* of the curve $x(s)$.

    Exercise 1. Prove that $T(s) = N(s) \times B(s)$, $N(s) = B(s) \times T(s)$.
  - Among all the planes containing the tangent line, the osculating plane is the one that "fits" the curve best. See Exercise 2 below.

1.2. Definition of torsion

- How quickly the osculating plane turns is clearly characterized by how quickly the unit binormal vector turns. We calculate

  $$B'(s) = T'(s) \times N(s) + T(s) \times N'(s) = T(s) \times N'(s). \tag{2}$$

  Now notice that $\|N(s)\| = 1 \Longrightarrow N'(s) \cdot N(s) = 0$. As $T(s) \cdot N(s) = 0$ too, we see that $B'(s) \parallel N(s)$. Consequently there is a scalar function $\tau(s)$ such that

  $$B'(s) = -\tau(s)\, N(s). \tag{3}$$

  We call $\tau(s)$ the *torsion* of the curve $x(s)$.
- Formula for $\tau(s)$. We calculate

  $$N'(s) = \left(\frac{x''(s)}{\kappa(s)}\right)' = \frac{x'''(s)}{\kappa(s)} - \frac{\kappa'(s)}{\kappa^2(s)}\, x''(s). \tag{4}$$

  Thus

  $$\begin{aligned} \tau(s) &= -(T(s) \times N'(s)) \cdot N(s) \\ &= -\left(\frac{T(s) \times x'''(s)}{\kappa(s)} - \frac{\kappa'(s)}{\kappa^2(s)}\, T(s) \times x''(s)\right) \cdot N(s) \\ &= -\frac{x'(s) \times x'''(s)}{\kappa(s)} \cdot N(s) \\ &= \frac{(x'(s) \times x''(s)) \cdot x'''(s)}{\kappa^2(s)}. \end{aligned} \tag{5}$$

Formula for torsion (arc length parametrization).

$$\tau(s) = \frac{(x'(s) \times x''(s)) \cdot x'''(s)}{\kappa^2(s)} \tag{6}$$

Remark 1. Alternative definition of torsion.

- Distance to osculating plane. We easily see that

$$d(s) = \|(x(s) - x(s_0)) \cdot B(s_0)\| \tag{7}$$

Exercise 2. Prove that, among all planes passing $x(s_0)$, the osculating plane is the only one satisfying

$$\lim_{s \to s_0} \frac{d(s)}{(s - s_0)^2} = 0. \tag{8}$$

- Taylor expansion of $x(s) - x(s_0)$ to order three:

$$x(s) - x(s_0) = x'(s_0)\,(s - s_0) + \frac{x''(s_0)}{2}\,(s - s_0)^2 + \frac{x'''(s_0)}{6}\,(s - s_0)^3 + R(s, s_0) \tag{9}$$

where $\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{(s - s_0)^3} = 0$.

- Substituting (9) into (7) we see that

$$\lim_{s \to s_0} \frac{d(s)}{(s - s_0)^3/6} = |x'''(s_0) \cdot B(s_0)|. \tag{10}$$

Exercise 3. Show that

$$|x'''(s_0) \cdot B(s_0)| = \left| \frac{(x'(s) \times x''(s)) \cdot x'''(s)}{\kappa(s)} \right|. \tag{11}$$

Compare to (6). Discuss possible reasons for the difference. Do you think (6) is a more reasonable definition for torsion? Why?

1.3. Examples

Example 2. The torsion of any plane curve is 0.

PROPOSITION 3. *Let $x(s)$ be a curve such that $\kappa > 0$ and $\tau = 0$ everywhere. Then $x(s)$ is a plane curve.*

Remark 4. Note that the assumption $\kappa > 0$ cannot be dropped.

Proof. We see that $\tau = 0 \Longrightarrow (x'(s) \times x''(s)) \cdot x'''(s) = 0$ (Note that we need $\kappa > 0$ here). Thus we see that there is a constant vector $v \in \mathbb{R}^3$ such that $x'(s) \cdot v = 0$ for all s. From this we have $(x(s) - x(s_0)) \cdot v = \int_{s_0}^{s} x'(t) \cdot v \, \mathrm{d}t = 0$ and the conclusion follows. $\square$

Example 5. We calculate the torsion of the curve $x(t) = \left(\frac{1}{\sqrt{3}}\cos t + \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{3}}\cos t, \frac{1}{\sqrt{3}}\cos t - \frac{1}{\sqrt{2}}\sin t\right)$. Note that we have seen there that t is already the arc length parameter.

$$x'(t) = \left(-\frac{1}{\sqrt{3}}\sin t + \frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{3}}\sin t, -\frac{1}{\sqrt{3}}\sin t - \frac{1}{\sqrt{2}}\cos t\right), \tag{12}$$

$$x''(t) = \left(-\frac{1}{\sqrt{3}}\cos t - \frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{3}}\cos t, -\frac{1}{\sqrt{3}}\cos t + \frac{1}{\sqrt{2}}\sin t\right) = -x(t). \tag{13}$$

Thus clearly

$$(x'(t) \times x''(t)) \cdot x'''(t) = (x'(t) \times x''(t)) \cdot (-x'(t)) = 0. \tag{14}$$

Thus we see that $x(t)$ is in fact a plane curve and is in fact a circle.

2. Frenet-Serret equations

One set of ODE determining the evolution of the vectors $T(s), N(s), B(s)$.

- We have seen that at a point $p = x(s_0)$ we could define three mutually orthogonal unit vectors:

$$T(s_0) := x'(s_0), \quad N(s_0) := \frac{x''(s_0)}{\kappa(s_0)}, \qquad B(s_0) := T(s_0) \times N(s_0). \tag{15}$$

  Since we are in $\mathbb{R}^3$, any other vector at p is a linear combination of these three vectors.

- We have also see that the evolution of these three vectors are governed by

$$T'(s) = \kappa(s)\, N(s), \qquad B'(s) = -\tau(s)\, N(s). \tag{16}$$

- For the equation for $N(s)$, we notice that

$$-\tau = (T \times N') \cdot N = (N \times T) \cdot N' = -B \cdot N' \tag{17}$$

  and

$$(T \cdot N)' = 0 \Longrightarrow N' \cdot T = -\kappa. \tag{18}$$

  Together with the fact that $N' \cdot N = 0$, we see that

$$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s). \tag{19}$$

Frenet-Serret equations for arc length parametrization.

$$\begin{array}{rcl} T' & = & \kappa N \\ N' & = & -\kappa T + \tau B \\ B' & = & -\tau N \end{array}. \tag{20}$$

Remark 6. It turns out that (20) completely determines the curve. More specifically we have

THEOREM 7. (THEOREM 2.3.6 OF TEXTBOOK) *Let* $\mathrm{K}(s)>0$ *and* $\mathrm{T}(s)$ *be given for all* $s\in(\alpha,\beta)$. *Let* x_0, T_0, N_0,s_0 *be given where* $x_0\in\mathbb{R}^3$, $T_0\perp N_0$ *are unit vectors, and* $s_0\in(\alpha,\beta)$. *Then there is a unique curve with* s *as its arc length parameter, satisfying* $x(s_0)=x_0$, *with* T_0, N_0 *as its unit tangent and normal vectors at* x_0, *and takes* $\mathrm{K}(s),\mathrm{T}(s)$ *as its curvature and torsion for* $s\in(\alpha,\beta)$.

Exercise 4. Can we drop the "with s as its arc length parameter" part?

3. Curvature and torsion in general parametrization

3.1. Formulas

- The key idea is that κ,τ,T,N,B should be independent of parametrization. In other words, if $x(t)$ and $x(s)$ are two parametrizations of the same curve, and $p=x(t_0)=x(s_0)$, then we must have $\kappa(t_0)=\kappa(s_0),\tau(t_0)=\tau(s_0)$, and so on.
- We will try to obtain the formulas intuitively here. For rigorous derivation, please see §2.3 of the textbook. In the following let $x(t)$ be a curve not necessarily in arc length parametrization.
  - T is the unit tangent vector. So we must have
  $$T(t)=\frac{x'(t)}{\|x'(t)\|}. \tag{21}$$
  - B. Since N clearly lies in the plane spanned by x' and x'', $B\parallel x'\times x''$. Consequently we must have
  $$B(t)=\frac{x'(t)\times x''(t)}{\|x'(t)\times x''(t)\|}. \tag{22}$$
  - N. We have
  $$N(t)=B(t)\times T(t)=\frac{(x'(t)\times x''(t))\times x'(t)}{\|x'(t)\times x''(t)\|\,\|x'(t)\|}. \tag{23}$$
  - κ. We have $\kappa=\frac{\mathrm{d}T}{\mathrm{d}s}\cdot N=\frac{1}{\|x'(t)\|}T'\cdot N$ which gives
  $$\kappa(t)=\frac{x''(t)\cdot[(x'(t)\times x''(t))\times x'(t)]}{\|x'(t)\|^3\,\|x'(t)\times x''(t)\|}=\frac{\|x'(t)\times x''(t)\|}{\|x'(t)\|^3}. \tag{24}$$
  - τ. We have $\tau=-\frac{\mathrm{d}B}{\mathrm{d}s}\cdot N$ which gives
  $$\begin{aligned}\tau &= -\frac{1}{\|x'(t)\|^2}\frac{(x'(t)\times x'''(t))\cdot[(x'(t)\times x''(t))\times x'(t)]}{\|x'(t)\times x''(t)\|^2}\\ &= -\frac{(x'(t)\times x'''(t))\cdot x''(t)}{\|x'(t)\times x''(t)\|^2}\\ &= \frac{(x'(t)\times x''(t))\cdot x'''(t)}{\|x'(t)\times x''(t)\|^2}. \end{aligned}\tag{25}$$

Exercise 5. In the above we have used the vector identity

$$(u \times v) \times w = -(v \cdot w)\, u + (u \cdot w)\, v \tag{26}$$

for $u, v, w \in \mathbb{R}^3$. Prove this identity and identify where it is used.

DG of Curves: Formulas for general parametrization

$$\kappa = \frac{\|x' \times x''\|}{\|x'\|^3}. \tag{27}$$

$$\tau = \frac{(x' \times x'') \cdot x'''}{\|x' \times x''\|^2}. \tag{28}$$

$$T = \frac{x'}{\|x'\|}, \tag{29}$$

$$B = \frac{x' \times x''}{\|x' \times x''\|}, \tag{30}$$

$$N = B \times T. \tag{31}$$

Warning

In exams, the formulas (27) and (28) will be provided, but (28–30) will not be provided.

Exercise 6. Does the Frenet-Serret equations

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned} \tag{32}$$

still hold for general parametrization?

Exercise 7. Show that if $x(t)$ is given by $(t, f(t))$ where $f\colon (\alpha, \beta) \mapsto \mathbb{R}^2$, then

$$\tau = \frac{\det\left(f'' \quad f''' \right)}{\|f''\|^2 + |\det\left(f'' \quad f''' \right)|^2}. \tag{33}$$

3.2. Examples

Example 8. Consider the curve $(\cos t, \sin t, 2t)$. We would like to calculate T, N, B, κ, τ.

- Preparation. We calculate

$$x'(t) = (-\sin t, \cos t, 2), \tag{34}$$

$$x''(t) = (-\cos t, -\sin t, 0), \tag{35}$$

$$x'''(t) = (\sin t, -\cos t, 0), \tag{36}$$

$$\|x'(t)\| = \sqrt{5}, \tag{37}$$

$$x'(t) \times x''(t) = (2\sin t, -2\cos t, 1), \tag{38}$$

$$\|x'(t) \times x''(t)\| = \sqrt{5}. \tag{39}$$

- Thus we have

$$\begin{aligned} T(t) &= \frac{x'}{\|x'\|} = \frac{1}{\sqrt{5}}(-\sin t, \cos t, 2), &(40)\\ B(t) &= \frac{x' \times x''}{\|x' \times x''\|} = \frac{1}{\sqrt{5}}(2\sin t, -2\cos t, 1), &(41)\\ N(t) &= B \times T = -(\cos t, \sin t), &(42)\\ \kappa(t) &= \frac{\|x' \times x''\|}{\|x'\|^3} = \frac{1}{5}, &(43)\\ \tau(t) &= \frac{(x' \times x'') \cdot x'''}{\|x' \times x''\|^2} = \frac{2}{5}. &(44) \end{aligned}$$

Exercise 8. Consider the curve $x(t) = (\cos t, \sin t, e^t)$. Without doing any calculation, can you predict the behavior of $\kappa(t), \tau(t)$ as $t \longrightarrow \infty$? Check your prediction through calculation.

Example 9. Consider the curve $y = f(x), z = 0$. We parametrize it as $x(t) = (t, f(t), 0)$. Then

$$\begin{aligned} x'(t) &= (1, f', 0), &(45)\\ x''(t) &= (0, f'', 0), &(46)\\ x'''(t) &= (0, f''', 0), &(47)\\ \|x'(t)\| &= \sqrt{1 + (f')^2}, &(48)\\ x'(t) \times x''(t) &= (0, 0, f''), &(49)\\ \|x'(t) \times x''(t)\| &= |f''|. &(50) \end{aligned}$$

Therefore

$$\begin{aligned} T(t) &= \frac{(1, f'(t), 0)}{\sqrt{1 + f'(t)^2}}, &(51)\\ B(t) &= (0, 0, \mathrm{sgn}(f''(t))), &(52)\\ N(t) &= \mathrm{sgn}(f''(t))\,(-f'(t), 1), &(53)\\ \kappa(t) &= \frac{\|x' \times x''\|}{\|x'\|^3} = \frac{|f''|}{\left(\sqrt{1 + (f')^2}\right)^3}, &(54)\\ \tau(t) &= 0. &(55) \end{aligned}$$

4. The local canonical form (optional)

- Let $x(s)$ be the arc length parametrization of the curve. We can calculate its Taylor expansion near s_0:

$$x(s) - x(s_0) = x'(s_0)\,(s - s_0) + \frac{x''(s_0)}{2}(s - s_0)^2 + \frac{x'''(s_0)}{6}(s - s_0)^3 + R(s, s_0) \qquad (56)$$

where $\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{|s - s_0|^3} = 0$.

Now we try to re-write (56) using $\kappa, \tau, T, N, B, s, s_0$ only. Clearly $x'(s_0) = T(s_0)$ and $x''(s_0) = \kappa(s_0)\,N(s_0)$. By (6) we have

$$x'''(s_0) \cdot B(s_0) = \kappa(s_0)\,\tau(s_0). \qquad (57)$$

On the other hand,

$$x'(s)\cdot x''(s)=0\Longrightarrow x'(s)\cdot x'''(s)=-x''(s)\cdot x''(s)\Longrightarrow x'''(s_0)\cdot T(s_0)=-\kappa(s_0)^2. \tag{58}$$

Finally $N(s)\cdot x''(s)=\kappa(s)$ gives

$$N'(s)\cdot x''(s)+N(s)\cdot x'''(s)=\kappa'(s_0)\Longrightarrow x'''(s_0)\cdot N(s_0)=\kappa'(s_0). \tag{59}$$

Summarizing, we have

$$x'''(s_0)=-\kappa(s_0)^2\,T(s_0)+\kappa'(s_0)\,N(s_0)+\kappa(s_0)\,\tau(s_0)\,B(s_0). \tag{60}$$

Therefore we have

$$x(s)-x(s_0)=a(s,s_0)\,T(s_0)+b(s,s_0)\,N(s_0)+c(s,s_0)\,B(s_0)+R(s,s_0) \tag{61}$$

where

$$a(s,s_0) = (s-s_0)-\frac{\kappa(s_0)^2}{6}(s-s_0)^3, \tag{62}$$

$$b(s,s_0) = \frac{\kappa(s_0)\,(s-s_0)^2}{2}+\frac{\kappa'(s_0)\,(s-s_0)^3}{6}, \tag{63}$$

$$c(s,s_0) = \frac{\kappa(s_0)\,\tau(s_0)\,(s-s_0)^3}{6}, \tag{64}$$

$$\lim_{s\to s_0}\frac{\|R(s,s_0)\|}{|s-s_0|^3} = 0. \tag{65}$$