Lecture 6: Differential Geometry of Curves I

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study how a curve curves. For simplicity we assume the curve is already in arc length parameter. We will show that the curving of a general curve can be characterized by two numbers, the curvature and the torsion.

The required textbook sections are: §2.1.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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Throughout this lecture we assume arc length parametrization.

1. Curvature

Curvature measures how quickly a curve turns, or more precisely how quickly the unit tangent vector turns.

1.1. The definition of the curvature

- Consider a curve $x(s): (\alpha, \beta) \mapsto \mathbb{R}^3$. Then the unit tangent vector of $x(s)$ is given by $x'(s)$. Consequently, how quickly $x'(s)$ turns can be characterized by the number
$$\kappa(s) := \|x''(s)\| \tag{1}$$
which call the curvature of the curve.
- As $\|x'(s)\| = 1$, we have $x''(s) \perp x'(s)$. This leads to the following definition.

Definition 1. *Let $x(s)$ be a curve parametrized by arc length. Then its curvature is defined as $\kappa(s) := \|x''(s)\|$. We further denote by $N(s)$ the unit vector $x''(s) / \|x''(s)\|$ and call it the normal vector at s. We also denote the unit tangent vector $x'(s)$ by $T(s)$.*

Formulas for curves in arc length parametrization.

- Curvature.
$$\kappa(s_0) = \|x''(s_0)\|. \tag{2}$$
- Tangent and Normal vectors.
$$T(s_0) = x'(s_0), \qquad N(s_0) = \frac{x''(s_0)}{\|x''(s_0)\|}. \tag{3}$$

1.2. Alternative characterization of the curvature (optional)

- Consider a curve $x(s)$: $(\alpha, \beta) \mapsto \mathbb{R}^3$. Let $p = x(s_0)$ for some $s_0 \in (\alpha, \beta)$. We try to understand how quickly $x(s)$ turns aways from the tangent line at $x(s_0)$.
- The equation for the tangent line is $x(s_0) + t\, x'(s_0)$.
- The distance from a point $x(s)$ to the tangent line is
$$d(s) := \|(x(s) - x(s_0)) \times x'(s_0)\| \tag{4}$$
Note that here we have used the fact that $x'(s_0)$ is a unit vector.
- Now recall Taylor expansion:
$$x(s) - x(s_0) = x'(s_0)\,(s - s_0) + \frac{1}{2} x''(s_0)\,(s - s_0)^2 + R(s, s_0) \tag{5}$$
where $\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{(s - s_0)^2} = 0$.

- Substituting (5) into (4), we see that

$$d(s) = \left\| \frac{(s-s_0)^2}{2} (x''(s_0) \times x'(s_0)) + R(s, s_0) \times x'(s_0) \right\| \tag{6}$$

and consequently

$$\lim_{s \to 0} \frac{d(s)}{(s-s_0)^2/2} = \|x''(s_0) \times x'(s_0)\| \tag{7}$$

Exercise 1. Prove (7).

- Thus we see that the quantity $\|x''(s_0) \times x'(s_0)\|$ measures how the curve "curves" at the point $x(s_0)$. We will denote it by $\kappa(s_0)$ and call it the curvature of the curve at $x(s_0)$.

Exercise 2. Prove that $\|x''(s_0) \times x'(s_0)\| = \|x''(s_0)\| = \kappa(s_0)$.

Exercise 3. (7) can be derived slightly differently as follows.

i. Find T such that $x(s) - [x(s_0) + T x'(s_0)] \perp x'(s_0)$.

ii. Then $d(s) = \|x(s) - [x(s_0) + T x'(s_0)]\|$.

iii. Calculate the limit $\lim \frac{d(s)}{(s-s_0)^2/2}$.

1.3. Examples

Example 2. For the unit circle, the curvature is constantly 1. For a circle with radius R, the curvature is constantly $1/R$.

Example 3. (Shifrin2016) Let $x(t) = \left(\frac{1}{\sqrt{3}} \cos t + \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t, \frac{1}{\sqrt{3}} \cos t - \frac{1}{\sqrt{2}} \sin t\right)$. We calculate

- Tangent vector:

$$x'(t) = \left(-\frac{1}{\sqrt{3}} \sin t + \frac{1}{\sqrt{2}} \cos t, -\frac{1}{\sqrt{3}} \sin t, -\frac{1}{\sqrt{3}} \sin t - \frac{1}{\sqrt{2}} \cos t\right) \tag{8}$$

- $\|x'(t)\| = 1$ so we are already in arc length parametrization.
- We have $\|x''(t)\| = 1$.

So the curvature of this curve is constantly 1.

Example 4. (Shifrin2016) Let $x(t) = \left(e^t, e^{-t}, \sqrt{2}\, t\right)$. We calculate

- Tangent vector:

$$x'(t) = \left(e^t, -e^{-t}, \sqrt{2}\right) \tag{9}$$

and therefore

- $\|x'(t)\| = e^t + e^{-t}$.
- To re-write it into arc length parametrization, we need to find a new parameter s such that $s = S(t)$ with

$$S'(t) = e^t + e^{-t} \tag{10}$$

as s would be the arc length parameter:

$$\|x'(t)\| = \|x'(s)\, S'(t)\| = |S'(t)|. \tag{11}$$

We see that $S(t) = e^t - e^{-t}$.

- Solve t as a function of s:

$$e^t = \frac{s + \sqrt{s^2+4}}{2}. \tag{12}$$

- Thus $x(s)$ is given by

$$\left(\frac{s+\sqrt{s^2+4}}{2}, \frac{\sqrt{s^2+4}-s}{2}, \sqrt{2}\ln\left(\frac{s+\sqrt{s^2+4}}{2}\right)\right). \tag{13}$$

- We calculate

$$x'(s) = \left(\frac{1}{2} + \frac{s}{2\sqrt{s^2+4}}, \frac{s}{2\sqrt{s^2+4}} - \frac{1}{2}, \frac{\sqrt{2}}{\sqrt{s^2+4}}\right). \tag{14}$$

To make sure our calculation is correct, we check

$$\|x'(s)\| = 1. \tag{15}$$

- Finally we calculate

$$x''(s) = \left(8\,(s^2+4)^{-3/2}, 8\,(s^2+4)^{-3/2}, -\sqrt{2}\, s\,(s^2+4)^{-3/2}\right) \tag{16}$$

which gives

$$\kappa(s) = \|x''(s)\| = \frac{\sqrt{2}\,\sqrt{s^2+64}}{\sqrt{s^2+4}^3}. \tag{17}$$

Remark 5. We see that with only (2), the calculation of curvatures could often be awkward due to the difficulty of explicitly writing down arc length parametrization. Thus we need a formula for curvature that works for general, not just arc length, parametrization. We will derive that in the next lecture.

PROPOSITION 6. *A plane curve with constant curvature is part of a circle/line.*

Proof. Let $x(s)$ be such that $\kappa(s)$ is constant.

- $\kappa = 0$. In this case we have $\ddot{x}(s) = 0$ which means $\dot{x}$ is constant.
- $\kappa > 0$. Let $y(s) := x(s) + \kappa^{-1} N(s)$. Then we calculate

$$y'(s) \;=\; x'(s) + \kappa^{-1} N'(s).$$

Clearly we have $y'(s) \cdot N(s) = 0$.

On the other hand, using $x'(s) \cdot N(s) = 0$ we see that

$$x'(s) \cdot n'(s) = -x''(s) \cdot N(s) = -\kappa. \tag{18}$$

Thus

$$y'(s) \cdot x'(s) = 1 + \kappa^{-1}(-\kappa) = 0. \tag{19}$$

Consequently $y'(s) \perp \{x'(s),\ N(s)\}$. As $x(s)$ is a plane curve, $y'(s)$ belongs to the plane spanned by $x'(s)$ and $N(s)$. Consequently $y'(s) = 0$ and therefore $y(s) = y_0$ is a constant. Now we see that

$$\|x(s) - y_0\| = \|-\kappa^{-1} N(s)\| = \kappa^{-1} \tag{20}$$

is a constant which means $x(s)$ is part of a circle. □

Remark 7. Proposition 6 ceases to hold if we drop the "plane curve" assumption.