Lecture 1: Introduction

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

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1. 348 viewed through development of mathematics

Math 348 studies the properties of curves and surfaces in $\mathbb{R}^3$, the three dimensional Euclidean space, using calculus as the main tool. This topic is also called "classical differential geometry", in contrast to "modern differential geometry" which studies more abstract geometrical objects (all kinds of "manifolds"–sets that can be equipped with a "differential structure"). Classical differential geometry was mainly developed during the 18th century, starting from the invention of calculus in the late 17th century and ending with the work of Carl Friedrich Gauss in the early 19th century.

1.1. Geometry before calculus

- Mathematics started with counting, believed to be about 20,000 years ago.[1]
- Geometry most likely started long after the agriculture revolution.
- The initial purpose of geometry is measuring fields.
    - The first geometrical objects are circles and rectangles. The former is easy to draw and the latter is the simplest shape that can tile a larger field.
    - From the need to obtain rectangles arised the need to obtain the right angle, and all kinds of "proto"-Pythagorean theorems.[2]
- Architecture also helped the development of geometry (pyramids, etc.).
- Field measuring/architecture only inspired very crude theory.
    - Precise results are not necessary. Approximation is enough.
    - The nature of technology dictates that no precise theory is needed. Large room for error is necessary for robustness.
- The true motivation behind most development of classical geometry:
    - Before Alexander the Great (classical period): The unique living environment and philosophy of ancient Greek citizens.
    - After Alexander the Great (Alexanderian period[3]): Astronomy.[4]
- Development of classical geometry.
    - Classical period.
        - Abstract.

$$\text{Area of circle} = \text{constant} \times r^2. \tag{1}$$

        - Pythagoreans; Euclid; Appollonius.
    - Alexandrian period.
        - More practical.

          What is the value of the constant in (1)?

1. See e.g. Ishango bone: https://en.wikipedia.org/wiki/Ishango_bone.
2. However note that the rigorous formulation and proof of "Pythagorean Theorem" is not necessary for this.
3. As well as other major cultures.
4. It is not possible to manipulate the movement of stars. Therefore the only thing left to do is understanding.

– Archimedes.

Exercise 1. Calculate the circumference and area of regular polygons of 6, 12, 24 sides inscribed in the unit circle, and obtain approximate values of π. Compare the values obtained from the circumference and the area. Which ones are better approximations? Can you explain why?

- The problem with classical (Euclidean) geometry.
  - Can only study circles and straight lines.[5]
    - This could be the reason why ancient Greeks are willing to work through tens of "epi-cycles" but not willing to make the (much simpler to us) assumption that the trajectories of planets are elliptical.[6]
  - The methods are ad hoc. Each problem requires a new idea to solve.

    "every proof in Euclidean geometry called for some new, often ingenious, approach." [7]
  - Has very limited ability dealing with problems in $\mathbb{R}^3$.

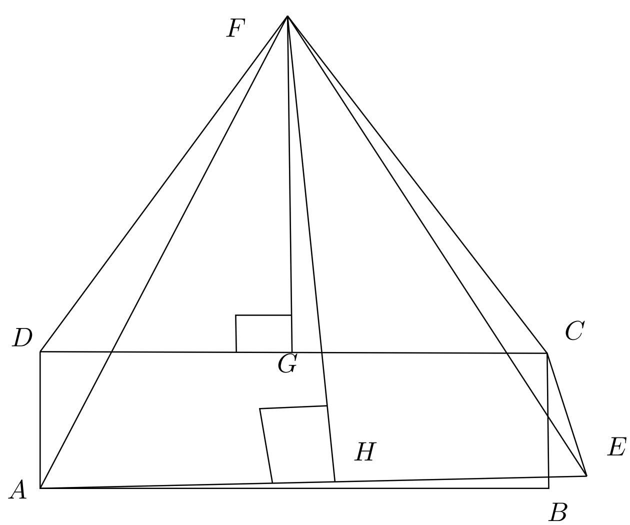
    Exercise 2. Prove by symmetry that

$$\text{Area of spherical triangle} = R^2 (\alpha + \beta + \gamma - \pi) \tag{2}$$

where R is the radius of the sphere, and α, β, γ are the three angles at the three vertices.

It is also easy to make (very hard to detect) mistakes in "everyday application" of classical geometry.

Exercise 3. Critique the following proof and "any angle is right angle".[8]



Here ABCD is a rectangle. CE = CB . G, H are midpoints of CD and AE. We see that $\triangle\text{FDA} = \triangle\text{FCE}$. Since $\angle\text{FDC} = \angle\text{FCD}$, we conclude $\angle\text{DCE} = \angle\text{CDA}$ is a right angle. [9]

5. The most complicated geometric objects in classical geometry are the conic curves, which is defined by intersecting a cone (obtained from a line and a circle) and a plane.

6. Another possible reason is related to the development of technology. At that time there is no accurate measurement of time. As a consequence, not much can be done unless the speed of the object is assume to be constant.

7. Morries Kline, "Mathematical thoughts from ancient to modern times". 1972.

8. Of course this is not really a problem with classical geometry.

- Help comes from
  - Algebra.
    - Algebra originates from counting.
    - Later the focus becomes solving algebraic equations arising from practical geometry.
    - Great progress has been made.
      - Al Khowarizmi.[10]
      - Omar Khayyam.[11]
      - Tartaglia, Cardano, Ferrari.[12]
  - Analysis.
    - Oresme.[13]
      - Primitive idea of "functions".
    - Fermat.
      - Represent curves by $(x, f(x))$.
  - Descartes.[14]
    - Analytic geometry (coordinate geometry).[15]
      1. Represent curves/surfaces by functions;
      2. Identify geometric relations with algebraic equations;
      3. Understand these equations through algebra.
- Spoilsports:
  - Lagrange, Ruffini, Abel, Galois.
  - Step 3 is often impossible, even for algebraic equations with one single unknown.
- Revolution.
  - Newton, Leibniz: Calculus.
    - Calculus: Originally means stones used for calculation in ancient Rome.[16]

9. The problem with the above proof is that FE is actually on the other side of C. To see this, consider first the case CE $\perp$ CB. In this case we see clearly that FE is above C. Now decrease the angle ECB. Assume there is $\theta_0 > 0$ such that FE is below C, then there must be an angle such that $C \in$ FE. But then as $D \notin$ FA, we have FA $<$ FD $+$ DA $=$ FC $+$ CE $=$ FE, contradiction.

10. https://en.wikipedia.org/wiki/Muhammad_ibn_Musa_al-Khwarizmi.

11. https://en.wikipedia.org/wiki/Omar_Khayyam.

12. https://en.wikipedia.org/wiki/Quartic_function#History.

13. https://en.wikipedia.org/wiki/Nicole_Oresme.

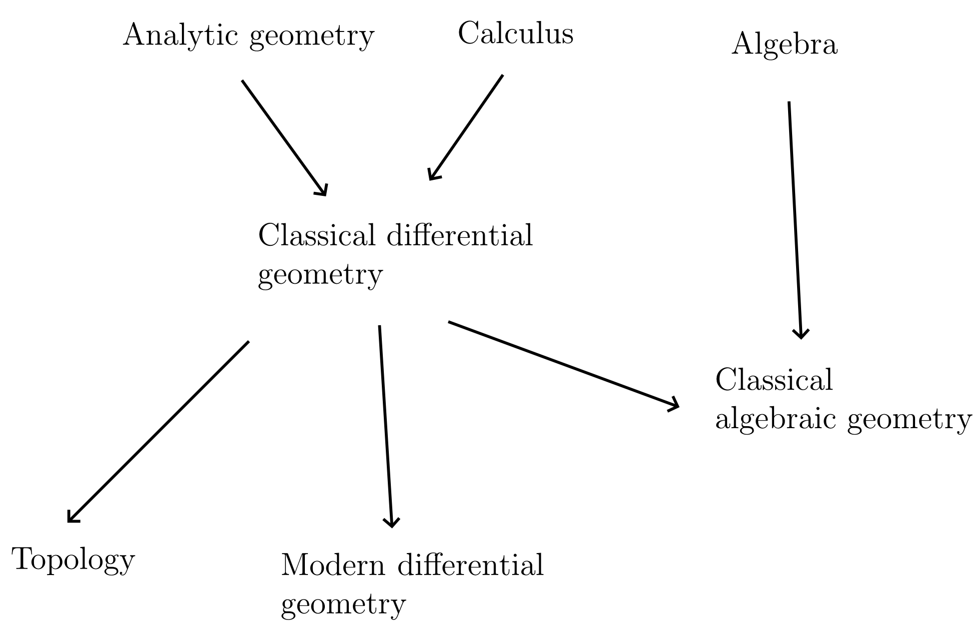
14. https://en.wikipedia.org/wiki/René_Descartes.

15. "analytic geometry" has nothing to do with analysis, which now roughly means rigorous treatment of calculus. The reason for this "misnomer" is that "analysis" originally means "argue backward from conclusion". For example, in algebra, the unknown is given a symbol and then manipulated "as if it is known". Thus algebra is a kind of "analysis".

  - A (very powerful) tool for calculation.
  - Easy to use.
    - Consists of a few formulas.
    - Can be applied to a wide range of problems.
    - The application of these formulas is mechanical, often "no understanding needed."

- Calculus makes it possible to do quantitative manipulation of arbitrary curves. Far beyond straight lines and circles.
- The application of calculus to geometric problems gave birth to classical differential geometry.
  1. Represent curves/surfaces by functions.
  2. Identify geometric relations with algebraic or differential equations.
  3. Understand these equations through calculus.
- Classical differential geometry is an "upgrade" of analytic geometry through calculus.

1.2. The development of classical differential geometry and its relations to other fields of mathematics



16. See e.g. https://en.wikipedia.org/wiki/Kidney_stone for a modern use of "calculus" to mean stone.

2. 348 for science and technology today

Although classical differential geometry is not "modern" and admittedly is unable to carry us through fancy theories like general relativity and string theory, it still has a great many applications in science and technology today.

2.1. Mechanics

- A mechanical system can be modelled as a set of objects whose movement are subject to constraints. The coordinates of these objects then "lives on" some surface in a (very) high dimensional Euclidean space $\mathbb{R}^n$.
- Similar idea can be applied to biology, for example to protein folding.

2.2. Image precessing

- Snake algorithm.[17] [18]
- Parametrization and deformation of surfaces.
- ...

3. How to study 348

Math 348 is in a sense unique in undergraduate mathematics courses, as it puts equal emphasis on concepts and calculation, on algebraic manipulation and geometric visualization, on seeing the big picture and working through calculations down to the last detail.

3.1. Prerequisites

From §1 we have seen that classical differential geometry applies multivariable calculus to geometric problems. The implication of this fact is two-fold.

- Multivariable calculus is about functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Among these functions, the simplest and most fundamental is the linear ones which could be represented by matrices. Therefore linear algebra is heavily involved.
- Just like Euclidean geometry/analytic geometry leads to algebraic equations, differential geometry leads to equations involving derivatives, that is differential equations.

Thus the natural prerequisites are

i. multivariable calculus;

ii. linear algebra;

iii. differential equations.

Note that we will only need basics from ii and iii.

17. https://en.wikipedia.org/wiki/Active_contour_model.
18. https://youtu.be/y3sVTrcxXwc.

3.2. Basic ideas of classical differential geometry and their implications to how to solve a DG problem

Classical DG can be viewed as a three-step procedure:

1. Obtain a mathematical representation of the geometric objects under study.
2. Apply (multivariable) calculus to these (now mathematical) objects.
3. Give geometrical interpretation to the results of the calculation.

Therefore, when facing a DG problem, one should adopt the following three-step approach:

1. Determine what to calculate;
2. Carry out the (could be very long and tedious) calculation;
3. Interpret the results.

3.3. Warnings and advices

- Steps 1 (determine what to calculate) and 3 (interpret the results) are usually very straightforward in most "easy" courses, such as calculus and differential equations.
- The Step 2 (calculation) is often straightforward in most "hard" courses, such as abstract algebra.
- All three steps are important in 348. Figuring out what to calculate gets only one third of the work done, as the calculation could be long and non-trivial, with obstacles here and there requring clever tricks to overcome.
- It is important to be patient and be willing to "get hands dirty".

4. Overview of the course

4.1. Review

- Two weeks; two homeworks.
- Curves and surfaces in multivariable calculus.
  - Representation;
  - Length of curves;
  - Area of surfaces.

4.2. Local theory of curves

- One week; one homework;
- How to quantitatively measure the "curving" of a curve in $\mathbb{R}^3$.

4.3. Local theory of surfaces

- Four weeks; four homeworks;
- Measuring distance along a surface;

- How to quantitatively measure the "curving" of a surface in $\mathbb{R}^3$.

4.4. Topics

- Three weeks; two homeworks;
- "global" properties of curves and surfaces.