Math 348 Fall 2016

Homework 8: Gauss and Codazzi Equations

(Total 20+5 pts; Due Nov. 25 12pm)

Question 1. (10 pts) *Decide whether there is a parametrized surface* $\sigma(u,v)$ *with*

a) (5 pts) $\mathbb{E}=\mathbb{G}=1, \mathbb{F}=0$ *and* $\mathbb{L}=\mathbb{N}=e^{2u}, \mathbb{M}=0$.

b) (5 pts) $\mathbb{E}=1, \mathbb{F}=0, \mathbb{G}=\sin^2 u,\ \mathbb{L}=\sin^2 u, \mathbb{M}=0, \mathbb{N}=1$.

Solution.

a) We first calculate the Christoffel symbols. Recall that

$$\begin{array}{rcl} \sigma_{uu} & = & \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L}\, N \\ \sigma_{uv} & = & \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M}\, N \, . \\ \sigma_{vv} & = & \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N}\, N \end{array} \tag{1}$$

From the first equation we have

$$\Gamma_{11}^1 = \Gamma_{11}^1 \sigma_u \cdot \sigma_u = \sigma_{uu} \cdot \sigma_u = \left(\frac{\mathbb{E}}{2} \right)_u = 0, \tag{2}$$

due to the fact that $\sigma_u \cdot \sigma_u = \mathbb{E} = 1, \sigma_u \cdot \sigma_v = \mathbb{F} = 0, \sigma_u \cdot N = 0$. Similarly we find that all other Γ_{ij}^k are zero too. The Gauss equations now yield $e^{4u} = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = K = 0$, contradiction. Therefore such surface does not exist.

b) Again we start from

$$\begin{array}{rcl} \sigma_{uu} & = & \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L}\, N \\ \sigma_{uv} & = & \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M}\, N \, . \\ \sigma_{vv} & = & \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N}\, N \end{array} \tag{3}$$

We solve

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0, \quad \Gamma_{12}^2 = \frac{\cos u}{\sin u}, \quad \Gamma_{22}^1 = -\sin u \cos u. \tag{4}$$

We also have

$$K = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = 1. \tag{5}$$

The Codazzi-Mainradi now become

$$\begin{array}{rcl} 0 & = & 0 \\ 0 & = & \sin^2 u\, (-\sin u \cos u) \end{array} \tag{6}$$

which is not satisfied. Therefore such surface does not exist.

Question 2. (10 pts) *Let* S *be a surface with first fundamental form* $u^2\, \mathrm{d}u^2 + \beta\, u^2\, \mathrm{d}v^2$ *for some* $\beta > 0$*, and second fundamental form* $A(u,v)\, \mathrm{d}u^2 + B(u,v)\, \mathrm{d}v^2$.

a) (5 pts) *Find* β*. Change to: Prove that* $A(u,v)\, B(u,v) = \beta$.

b) (5 pts) *Prove that* $A(u,v), B(u,v)$ *are functions of* u *only.*

Proof. We first calculate the Christoffel symbols. We have

$$\sigma_{uu} = \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + A(u,v)\, N \Longrightarrow \begin{cases} \mathbb{E}\,\Gamma_{11}^1 = \sigma_{uu}\cdot\sigma_u \Longrightarrow \Gamma_{11}^1 = \dfrac{1}{u}, \\ \mathbb{G}\,\Gamma_{11}^2 = \sigma_{uu}\cdot\sigma_v \Longrightarrow \Gamma_{11}^2 = 0. \end{cases} \tag{7}$$

$$\sigma_{uv} = \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v \Longrightarrow \begin{cases} \mathbb{E}\,\Gamma_{12}^1 = \sigma_{uv}\cdot\sigma_u \Longrightarrow \Gamma_{12}^1 = 0, \\ \mathbb{G}\,\Gamma_{12}^2 = \sigma_{uv}\cdot\sigma_v \Longrightarrow \Gamma_{12}^2 = \dfrac{1}{u}. \end{cases} \tag{8}$$

$$\sigma_{vv} = \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + B(u,v)\, N \Longrightarrow \begin{cases} \mathbb{E}\,\Gamma_{22}^1 = \sigma_{vv}\cdot\sigma_u \Longrightarrow \Gamma_{22}^1 = -\dfrac{\beta}{u}, \\ \mathbb{G}\,\Gamma_{22}^2 = \sigma_{vv}\cdot\sigma_v \Longrightarrow \Gamma_{22}^2 = 0. \end{cases} \tag{9}$$

The Codazzi-Mainradi equations and the Gauss equations now become

$$A_v = 0, \tag{10}$$

$$-B_u = -\frac{A(u,v)}{u} - \frac{B(u,v)}{u}, \tag{11}$$

and

$$u^2 K = \frac{1}{u^2}, \tag{12}$$

$$0 = 0, \tag{13}$$

$$= 0, \tag{14}$$

$$u^2 K = \frac{1}{u^2}. \tag{15}$$

a) By the Gauss equations $u^2 K = \frac{1}{u^2} \Longrightarrow K = \frac{1}{u^4}$. On the other hand $K = \frac{A(u,v)\, B(u,v)}{\beta\, u^4}$. Therefore $A(u,v)\, B(u,v) = \beta$.

b) Due to (10) A is a function of u only. Since $A\, B = \beta$, we have $B = \beta / A$ is also a function of u only. □

The following are more abstract or technical questions. They carry bonus points.

QUESTION 3. (5 PTS) *Let S be such that $\kappa_1 \neq \kappa_2$ are both constants. Prove that its Gaussian curvature is* 0.

(You can assume that the surface patch is such that $\sigma_u \parallel t_1$, $\sigma_v \parallel t_2$ where t_1, t_2 are the principal vectors.)

Proof. With such $\sigma(u,v)$ we have the first and second fundamental forms to be $\mathbb{E}\, du^2 + \mathbb{G}\, dv^2$ and $\mathbb{L}\, du^2 + \mathbb{N}\, dv^2$, and furthermore $\kappa_1 = \frac{\mathbb{L}}{\mathbb{E}}$, $\kappa_2 = \frac{\mathbb{N}}{\mathbb{G}}$.

Now we calculate

$$\begin{aligned} \Gamma_{11}^1 &= \frac{\mathbb{E}_u}{2\,\mathbb{E}}, & \Gamma_{11}^2 &= -\frac{\mathbb{E}_v}{2\,\mathbb{G}}, \\ \Gamma_{12}^1 &= \frac{\mathbb{E}_v}{2\,\mathbb{E}}, & \Gamma_{12}^2 &= \frac{\mathbb{G}_u}{2\,\mathbb{G}}, \\ \Gamma_{22}^1 &= -\frac{\mathbb{G}_u}{2\,\mathbb{E}}, & \Gamma_{22}^2 &= \frac{\mathbb{G}_v}{2\,\mathbb{G}}. \end{aligned} \tag{16}$$

The Codazzi-Mainradi equations now become

$$\mathbb{L}_v = \frac{1}{2}\,\mathbb{E}_v\left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}}\right) = \frac{\kappa_1 + \kappa_2}{2}\,\mathbb{E}_v, \qquad \mathbb{N}_u = \frac{1}{2}\,\mathbb{G}_u\left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}}\right) = \frac{\kappa_1 + \kappa_2}{2}\,\mathbb{G}_u. \tag{17}$$

Together with $\mathbb{L}_v = \kappa_1\,\mathbb{E}_v, \mathbb{N}_u = \kappa_2\,\mathbb{G}_u$ and $\kappa_1 \neq \kappa_2$, we see that

$$\mathbb{L}_v = \mathbb{E}_v = \mathbb{N}_u = \mathbb{G}_u = 0. \tag{18}$$

Substituting (18) into (16) we see that

$$\Gamma_{11}^1 = \frac{\mathbb{E}_u}{2\,\mathbb{E}}, \qquad \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0, \qquad \Gamma_{22}^2 = \frac{\mathbb{G}_v}{2\,\mathbb{G}}. \tag{19}$$

Now the first Gauss equation becomes $\mathbb{E}\,K = 0$ and consequently $K = 0$. □