Homework 7: Parallel Transport and Geodesics

(Total 20 pts; Due Nov. 7 12pm)

Question 1. (5 pts) *Let γ be a curve on S. Let w be a tangent vector field parallel along γ. Find all $\lambda\colon \gamma \mapsto \mathbb{R}$ such that $\lambda\, w$ is still parallel along γ.*

Solution. Such λ's are constant functions.

We have $(\lambda\, w)' = \lambda\, w' + \lambda'\, w$. Now $\nabla_\gamma(\lambda w) = 0 \Longleftrightarrow (\lambda\, w)' \parallel N$. Since $w' \parallel N$, there must hold $\lambda'\, w \parallel N$, or equivalently $\lambda' = 0$.

On the other hand, if $\lambda = \text{constant}$ clearly $\nabla_\gamma w \Longrightarrow \nabla_\lambda(\lambda\, w) = 0$.

Question 2. (5 pts) *Let γ be a curve on S. Let $w, \tilde{w}$ be unit vector fields along γ. Further assume that at every $p \in \gamma$, there holds the angle between $w, \tilde{w}$, $\angle(w, \tilde{w}) = \theta_0$, a constant. Prove or disprove: w is parallel along γ if and only if $\tilde{w}$ is parallel along γ.*

Solution. The claim is true. We parametrize γ by some $x(t)$ and simply write $w(t), \tilde{w}(t)$. We discuss two cases.

1. $\angle(w, \tilde{w}) = 0$ or π. Then $\tilde{w} = w$ or $-w$. Clearly $\nabla_\gamma \tilde{w} = 0$.
2. Otherwise. Notice that this means $\{w,\ \tilde{w}\}$ for a basis for the tangent plane. By assumption we have $w \cdot \tilde{w} = \text{constant}$. Therefore
$$w' \cdot \tilde{w} + w \cdot \tilde{w}' = 0. \tag{1}$$
Since $\nabla_\gamma w = 0$, we have $w' \perp \tilde{w}$. Therefore $\tilde{w}' \cdot w = 0$. On the other hand, as $\|\tilde{w}\| = 1$ we have $\tilde{w}' \cdot \tilde{w} = 0$. Thus $\tilde{w} \parallel N$ and consequently $\nabla_\gamma \tilde{w} = 0$.

Question 3. (10 pts) *Let S be a surface parametrized by $\sigma(u, v) = (u, v, u\, v)$.*

a) (7 pts) *Calculate the Christoffel symbols $\Gamma_{11}^1, \ldots, \Gamma_{22}^2$.*

b) (2 pts) *Write down the geodesic equations for this surface.*

c) (1 pt) *Prove that $u = \text{constant}$ and $v = \text{constant}$ are geodesics.*

Proof.

a) We calculate
$$\sigma_u = (1, 0, v), \qquad \sigma_v = (0, 1, u) \tag{2}$$
which gives
$$\mathbb{E} = 1 + v^2, \qquad \mathbb{F} = u\, v, \qquad \mathbb{G} = 1 + u^2. \tag{3}$$
Consequently
$$\begin{aligned}
\Gamma_{11}^1 &= \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \quad \Gamma_{11}^2 = \frac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0,\\
\Gamma_{12}^1 &= \frac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{v}{1 + u^2 + v^2}, \quad \Gamma_{12}^2 = \frac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{u}{1 + u^2 + v^2},\\
\Gamma_{22}^1 &= \frac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \quad \Gamma_{22}^2 = \frac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0.
\end{aligned} \tag{4}$$

b) The geodesic equations are

$$u'' + \frac{2\,v}{1+u^2+v^2}\,u'\,v' \;=\; 0, \tag{5}$$

$$v'' + \frac{2\,u}{1+u^2+v^2}\,u'\,v' \;=\; 0. \tag{6}$$

c) For $u = u_0$, we take the parametrization $\sigma(u_0, t)$. Then we see that $u(t) = u_0$, $v(t) = t$ satisfy the above equations. Similarly we prove that $v =$ constant are geodesics.

Alternatively, we can prove this by noticing that $u =$ constant and $v =$ constant are straight lines. Therefore must be geodesics. □

The following are more abstract or technical questions. They carry bonus points.

There is no bonus question for this homework.