Homework 6: Curvatures for Surfaces

(Total 20 pts + bonus 5 pts; Due Oct. 28 12pm)

Question 1. (10 pts) *Calculate* H, K, κ_1, κ_2, t_1, t_2 *at the point* $(1, 1, 1)$ *for the surface* $z = x\,y$.

Question 2. (5 pts) *Let* $\sigma(u, v)$ *be a surface patch. Assume that* $\mathbb{E} = \mathbb{G}, \mathbb{F} = 0$.

a) (3 pts) *Prove that* $\sigma_{uu} + \sigma_{vv}$ *is perpendicular to* σ_u *and* σ_v.

b) (2 pts) *Prove that if* $\sigma_{uu} + \sigma_{vv} = 0$ *then* $H = 0$.

Question 3. (5 pts) *Let a surface* S *be such that* $H = 0, K \neq 0$. *Prove that its Gauss map* $\mathcal{G}$ *is conformal. That is if* σ *is a surface patch for* S *and we take* $\tilde{\sigma} := \mathcal{G} \circ \sigma$ *as the surface patch for* $\mathbb{S}^2$, *then there is a scalar function* λ *such that* $\tilde{\mathbb{E}} = \lambda\,\mathbb{E}, \tilde{\mathbb{F}} = \lambda\,\mathbb{F}, \tilde{\mathbb{G}} = \lambda\,\mathbb{G}$.

(Hint: Write $-N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v$, $-N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v$ *and recall the relations between* a_{ij} *and* $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}, H, K$. *It also helps to write things in matrix form.)*

The following are more abstract or technical questions. They carry bonus points.

Question 4. (**Bonus, 5 pts**) *Let* S *be a surface and* $p_0 \in S$. *Assume that there is a surface patch* σ *covering* $p_0 = \sigma(u_0, v_0)$ *such that at* p_0 *there holds* $\mathbb{E} = \mathbb{G} = 1$, $\mathbb{F} = 0$ *and* $\mathbb{E}_u = \mathbb{E}_v = \mathbb{F}_u = \mathbb{F}_v = \mathbb{G}_u = \mathbb{G}_v = 0$.

a) (3 pts) *Prove that the Gaussian curvature at* p *is*

$$K = \frac{\partial^2 \mathbb{F}}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 \mathbb{G}}{\partial u^2} - \frac{1}{2}\frac{\partial^2 \mathbb{E}}{\partial v^2}. \tag{1}$$

b) (1 pt) *Let* $\tilde{S}$ *be another surface such that there is a local isometry* $f: S \mapsto \tilde{S}$. *Prove that the Gaussian curvature at* $f(p)$ *is* $\tilde{K} = K$.

c) (1 pt) *Prove or disprove:* $\tilde{H} = H$.