Homework 5: The Second Fundamental Form

(Total 20 pts + bonus 5 pts; Due Oct. 14 12pm)

Question 1. (5 pts) *Calculate the second fundamental form of the surface*

$$\sigma(u,v)=(u,v,u\,v). \tag{1}$$

Question 2. (5 pts) *Let $f\colon S_1 \mapsto S_2$ be a local isometry. Let σ_1 be a surface patch for S_1 and let $\sigma_2 = f\circ\sigma_1$. Prove or disprove: There hold $\mathbb{L}_2=\mathbb{L}_1$, $\mathbb{M}_2=\mathbb{M}_1$, $\mathbb{N}_2=\mathbb{N}_1$.*

Question 3. (10 pts) *Let $x(s)=\sigma(u(s),v(s))$ be an arc length parametrized curve on a surface patch σ. Assume that at every s the curvature of $x(s)$ equals $|\kappa_n|$, the absolute value of the normal curvature of the surface at $x(s)$ along $x'(s)$.*

a) (5 pts) *Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,u'+\mathbb{F}\,v') = \frac{1}{2}(\mathbb{E}_u\,(u')^2+2\,\mathbb{F}_u\,u'\,v'+\mathbb{G}_u\,(v')^2), \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\,u'+\mathbb{G}\,v') = \frac{1}{2}(\mathbb{E}_v\,(u')^2+2\,\mathbb{F}_v\,u'\,v'+\mathbb{G}_v\,(v')^2). \tag{3}$$

b) (5 pts) *Let S be the cylinder $\sigma(u,v)=(\cos u,\sin u,v)$. Find all the curves $x(s)$ on S satisfying $\kappa(s)=|\kappa_n(s)|$ for every s. Note that to solve b) you don't need to know how to prove a).*

The following are more abstract or technical questions. They carry bonus points.

Question 4. (**Bonus, 5 pts**) *A normal section of a surface S is the intersection between S and a plane Π that is perpendicular to the tangent plane of the surface at every point of this intersection curve. Assume that at every $p\in S$, for every $w\in T_pS$, the plane spanned by the surface normal N and w intersects S along a normal section of S. Prove that S is part of a sphere.*