Solutions for Homework 3: Differential Geometry of Curves

(Total 20 pts + bonus 5 pts; Due Sept. 30 12pm)

Question 1. (10 pts) *Calculate T, N, B, κ, τ of the curve $x(t) = (t, t^2, t^4)$ at the point* $(1, 1, 1)$.

Solution. We have

$$x'(t) = (1, 2\,t, 4\,t^3), \tag{1}$$

$$x''(t) = (0, 2, 12\,t^2), \tag{2}$$

$$x'''(t) = (0, 0, 24\,t), \tag{3}$$

$$x'(t) \times x''(t) = (16\,t^3, -12\,t^2, 2), \tag{4}$$

$$(x'(t) \times x''(t)) \cdot x'''(t) = 48\,t, \tag{5}$$

$$\|x'(t)\| = \sqrt{1 + 4\,t^2 + 16\,t^6}, \tag{6}$$

$$\|x'(t) \times x''(t)\| = \sqrt{4 + t^4 + 256\,t^6}. \tag{7}$$

Therefore we have

$$T(t) = \frac{x'(t)}{\|x'(t)\|} = \frac{(1, 2\,t, 4\,t^3)}{\sqrt{1 + 4\,t^2 + 16\,t^6}} \Longrightarrow T(1) = \frac{(1, 2, 4)}{\sqrt{21}}, \tag{8}$$

$$B(t) = \frac{x'(t) \times x''(t)}{\|x'(t) \times x''(t)\|} = \frac{(16\,t^3, -12\,t^2, 2)}{\sqrt{4 + 144\,t^4 + 256\,t^6}} \Longrightarrow B(1) = \frac{(8, -6, 1)}{\sqrt{101}}, \tag{9}$$

$$N(1) = B(1) \times T(1) = \frac{(-26, -31, 22)}{\sqrt{2121}}, \tag{10}$$

$$\kappa(t) = \frac{\|x'(t) \times x''(t)\|}{\|x'(t)\|^3} = \frac{\sqrt{4 + 144\,t^4 + 256\,t^6}}{\left(\sqrt{1 + 4\,t^2 + 16\,t^6}\right)^3} \Longrightarrow \kappa(1) = \frac{2\sqrt{101}}{21\sqrt{21}}, \tag{11}$$

$$\tau(t) = \frac{(x'(t) \times x''(t)) \cdot x'''(t)}{\|x'(t) \times x''(t)\|^2} = \frac{48\,t}{4 + 144\,t^4 + 256\,t^6} \Longrightarrow \tau(t) = \frac{12}{101}. \tag{12}$$

QUESTION 2. (5 PTS) *Let f be a smooth function. Calculate the curvature and the torsion of the curve that is the intersection of $x = y$ and $z = f(x)$.*

Solution.

First we write down the parametrized curve:

$$x(t) = (t, t, f(t)). \tag{13}$$

Now we can calculate

$$x'(t) = (1, 1, f'(t)), \tag{14}$$

$$x''(t) = (0, 0, f''(t)), \tag{15}$$

$$x'''(t) = (0, 0, f'''(t)), \tag{16}$$

$$x'(t) \times x''(t) = (f''(t), -f''(t), 0), \tag{17}$$

$$(x'(t) \times x''(t)) \cdot x'''(t) = 0. \tag{18}$$

$$\|x'(t)\| = \sqrt{2 + (f'(t))^2}, \tag{19}$$

$$\|x'(t) \times x''(t)\| = \sqrt{2}\,|f''(t)|. \tag{20}$$

Therefore

$$\kappa(t) = \frac{\|x'(t) \times x''(t)\|}{\|x'(t)\|^3} = \frac{\sqrt{2}\,|f''(t)|}{\left(\sqrt{2 + (f'(t))^2}\right)^3}, \qquad \tau(t) = \frac{(x'(t) \times x''(t)) \cdot x'''(t)}{\|x'(t) \times x''(t)\|^2} = 0. \tag{21}$$

QUESTION 3. (5 PTS) *Let $x(s)$ be a curve with arc length parametrization, and satisfies $\|x(s)\| \leqslant \|x(s_0)\| \leqslant 1$ for all s sufficiently close to x_0. Prove $\kappa(s_0) \geqslant 1$. (Hint: Consider $f(s) = \|x(s)\|^2$. Then $f(s)$ has a local maximum at s_0. Calculate $f''(s_0)$)*

Proof. As $f(s) := \|x(s)\|^2 = x(s) \cdot x(s)$ reaches a local maximum at s_0, there holds $f''(s_0) \leqslant 0$. We calculate

$$\begin{aligned} 0 &\geqslant f''(s_0) \\ &= 2\, x''(s_0) \cdot x(s_0) + 2\, x'(s_0) \cdot x'(s_0) = 2\, [1 + \kappa(s_0)\, N(s_0) \cdot x(s_0)]. \end{aligned} \tag{22}$$

Thus we have

$$[N(s_0) \cdot x(s_0)]\, \kappa(s_0) \leqslant -1. \tag{23}$$

Since by definition $\kappa(s_0) \geqslant 0$, there must hold $[N(s_0) \cdot x(s_0)] < 0$. Consequently we have

$$\kappa(s_0) \geqslant \frac{-1}{N(s_0) \cdot x(s_0)} = \frac{-1}{-|N(s_0) \cdot x(s_0)|} = \frac{1}{|N(s_0) \cdot x(s_0)|}. \tag{24}$$

Finally, notice that as $\|x(s_0)\| \leqslant 1$, $\|N(s_0)\| = 1$, we must have $|N(s_0) \cdot x(s_0)| \leqslant 1$ and the conclusion follows. □

The following are more abstract or technical questions. They carry bonus points.

QUESTION 4. (**BONUS, 5 PTS**) *Let $x(t)$ be a smooth* ***plane curve****. Assume that the chord length between $x(t_1), x(t_2)$ depends only on $|t_1 - t_2|$ for all $t_1, t_2 \in (\alpha, \beta)$, that is there is some function F such that $\|x(t_1) - x(t_2)\| = F(|t_1 - t_2|)$ for all $t_1, t_2 \in (\alpha, \beta)$. Prove that $x(t)$ is part of either a circle or a straightline. (Hint: First from $\|x(t+\delta t) - x(t)\| = F(\delta t)$ show that $\|x'(t)\| = constant$ for every t. Next apply Taylor expansion to $\|x(t+\delta t) - x(t)\|^2 = F(\delta t)^2$ to reach the conclusion.)*

Proof. By assumption we have

$$\|x'(t)\| = \left\| \lim_{\delta t \to 0+} \frac{x(t+\delta t) - x(t)}{\delta t} \right\| = \lim_{\delta t \to 0+} \frac{\|x(t+\delta t) - x(t)\|}{\delta t} = \lim_{\delta t \to 0+} \frac{F(\delta t)}{\delta t} = F'(0), \tag{25}$$

note that clearly we must have $F(0) = 0$. Therefore $\|x'(t)\|$ is a constant and through setting $s = F'(0)\, t$ we can assume that the curve is parametrized by arc length. In the following we will write $x(s)$ instead of $x(t)$.

Now we have for every s and every $t > 0$,

$$F(t)^2 = (x(s+t) - x(s)) \cdot (x(s+t) - x(s)). \tag{26}$$

We Taylor expand $x(s+t)$ to order t^3:

$$x(s+t) - x(s) = x'(s)\, t + x''(s)\, \frac{t^2}{2} + x'''(s)\, \frac{t^3}{6} + R(s,t) \tag{27}$$

where $\lim_{t \to 0} \frac{\|R(s,t)\|}{t^3} = 0$ and substitute into (26),

$$F(t)^2 = t^2 + \|x''(s)\|^2\, \frac{t^4}{4} + \frac{x'(s) \cdot x'''(s)}{3}\, t^4 + \tilde{R}(s,t) \tag{28}$$

where $\lim_{t \to 0} \frac{\|\tilde{R}(s,t)\|}{t^4} = 0$.

Next we notice that

$$x'(s) \cdot x''(s) = 0 \Longrightarrow x'(s) \cdot x'''(s) = -\|x''(s)\|^2 = \kappa(s)^2. \tag{29}$$

Consequently we have

$$F(t)^2 = t^2 - \frac{\kappa(s)^2}{12}\, t^4 + \tilde{R}(s,t). \tag{30}$$

Which gives

$$\kappa(s)^2 = \frac{12}{t^4}\, [F(t)^2 - t^2 + \tilde{R}(s,t)]. \tag{31}$$

Taking limit $t \to 0+$ we see that

$$\kappa(s)^2 = \lim_{t \to 0+} \frac{12}{t^4}\, [F(t)^2 - t^2] \tag{32}$$

is independent of s, that is, is a constant. As the curve is a plane curve, we see that it must be a circle. □