Solutions to Homework 2: Basics of Curves and Surfaces

(Total 20 pts + bonus pts; Due Sept. 23 12pm)

Question 1. (5 pts) *Let $a, b, c > 0$.*

a) (2 pts) *Sketch the curve*

$$x(t) = \left(3\cos\frac{t}{c}, 3\sin\frac{t}{c}, \frac{4}{c}\,t\right). \tag{1}$$

b) (3 pts) *Find a value of c such that t is the arc length parameter in (1). Justify your claim.*

Solution.

a) See for example the "Parametric curve plotter" of Wolfram|Alpha Widgets.[1]

b) We calculate

$$x'(t) = \left(-\frac{3}{c}\sin\frac{t}{c}, \frac{3}{c}\cos\frac{t}{c}, \frac{4}{c}\right) \Longrightarrow \|x'(t)\| = \frac{5}{c}. \tag{2}$$

Therefore $c = 5$ is the value that makes t the arc length parameter.

1. Just search "wolfram alpha parametric curve plotter" in google.

QUESTION 2. (5 PTS) *Consider the hypoboloid* $x_3^2 = x_1^2 + x_2^2 + 2$. *Calculate its unit normal vectors at the point* $(1,1,2)$. *Note that there are two unit normal vectors at each point.*

Solution. First we write down a surface patch covering the point $(1,1,2)$:

$$\sigma(u,v) = \left(u, v, \sqrt{u^2+v^2+2}\right), \tag{3}$$

with $U = \mathbb{R}^2$, and $(1,1,2) = \sigma(1,1)$.

Thus we calculate

$$\sigma_u(1,1) = \left(1, 0, \frac{1}{2}\right), \qquad \sigma_v(1,1) = \left(0, 1, \frac{1}{2}\right). \tag{4}$$

Therefore

$$N(1,1,2) = \pm \frac{\sigma_u(1,1) \times \sigma_v(1,1)}{\|\sigma_u(1,1) \times \sigma_v(1,1)\|} = \pm \frac{\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)}{\left\|\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)\right\|} = \pm \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right). \tag{5}$$

QUESTION 3. (5 PTS) *Calculate the surface area of the torus in Exercise 4.2.5 in the textbook.*

Solution. In Exercise 4.2.5 the parametrization of the torus is given by $(0 < a < b)$

$$((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta) \tag{6}$$

for $0 \leqslant \theta, \varphi < 2\pi$. Note that this is not a surface patch as $0 \leqslant \theta, \varphi < 2\pi$ is not open in $\mathbb{R}^2$. To calculate the surface area, we consider the following surface patch

$$\sigma(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta), \qquad 0 < \theta, \varphi < 2\pi. \tag{7}$$

This patch covers everywhere of the torus except along the two circles $\theta = 0$ and $\varphi = 0$. As these two circles have area 0, we have the surface area to be

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^{2\pi} \|\sigma_\theta \times \sigma_\varphi\| \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= \int_0^{2\pi} \int_0^{2\pi} b\,(a + b\cos\theta) \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= 4\pi^2 a\, b. \end{aligned} \tag{8}$$

QUESTION 4. (5 PTS) *Let S be the surface $x_3 = x_1^2 + x_2^2$. Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$. Let $N(p)$ be the unit normal vector at $p \in S$, pointing down.*

a) (1 PTS) *Calculate $N(p_0)$ for $p_0 = (0,0,0)$.*

b) (2 PTS) *Write down a surface patch σ for S covering p_0, and a surface patch $\tilde{\sigma}$ for $\mathbb{S}^2$ covering $N(p_0)$.*

c) (2 PTS) *Calculate the matrix representation of the differential $D_{p_0} N: T_{p_0} S \mapsto T_{N(p_0)} \mathbb{S}^2$ with respect to the two surface patches you have just specified.*

Solution.

a) We see that S can be covered by a single surface patch $\sigma: \mathbb{R}^2 \mapsto \mathbb{R}^3$, $\sigma(u,v) = (u, v, u^2 + v^2)$. We have $p_0 = \sigma(0,0)$. Calculate

$$\sigma_u(0,0) = (1,0,0), \qquad \sigma_v(0,0) = (0,1,0) \tag{9}$$

therefore

$$N(p) = (0,0,-1). \tag{10}$$

Note that we have chosen the sign to make $N(p)$ pointing downward.

b) We already have σ. For $\tilde{\sigma}$ we need a surface patch for the unit sphere covering $(0,0,-1)$. One such patch is

$$\tilde{\sigma}: \{(u,v) | \, u^2 + v^2 < 1\} \mapsto \mathbb{R}^3, \qquad \tilde{\sigma}(u,v) = \left(u, v, -\sqrt{1 - u^2 - v^2}\right). \tag{11}$$

c) We have

$$(\tilde{\sigma})^{-1}(x,y,z) = (x,y). \tag{12}$$

d) We have (note that $N \circ \sigma(0,0)$ should point downward)

$$N \circ \sigma(u,v) = N(u, v, u^2 + v^2) = -\frac{(1,0,2\,u) \times (0,1,2\,v)}{\|(1,0,2\,u) \times (0,1,2\,v)\|} = \frac{(2\,u, 2\,v, 1)}{\sqrt{1 + 4\,u^2 + 4\,v^2}} \tag{13}$$

and therefore

$$F(u,v) := (\tilde{\sigma})^{-1} \circ N \circ \sigma(u,v) = \frac{(2\,u, 2\,v)}{\sqrt{1 + 4\,u^2 + 4\,v^2}}. \tag{14}$$

e) Finally we calculate

$$DF(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{15}$$

The following are more abstract or technical questions. They carry bonus points.

Question 5. (**Bonus, 5 pts**) *Let S be the torus described in Exercise 4.2.5 in the textbook. Find an atlas of surface patches for S. You need to write down explicit formulas, including the domain U, for each patch.*

Solution. We take

$$\sigma_1(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta) \tag{16}$$

with $U_1 = \{(\theta, \varphi): 0 < \theta, \varphi < 2\pi\}$;

$$\sigma_2(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta) \tag{17}$$

with $U_2 = \{(\theta, \varphi): 0 < \theta < 2\pi, -\pi < \varphi < \pi\}$;

$$\sigma_3(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta) \tag{18}$$

with $U_3 = \{(\theta, \varphi): -\pi < \theta < \pi, 0 < \varphi < 2\pi\}$;

$$\sigma_4(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta) \tag{19}$$

with $U_4 = \{(\theta, \varphi): -\pi < \theta < \pi, -\pi < \varphi < \pi\}$.