Solutions to Homework 1: Prerequisites

(Total 20 pts + bonus pts; Due Sept. 16 12pm)

Question 1. (5 pts) *Let* $f, g\colon \mathbb{R} \mapsto \mathbb{R}^3$ *be differentiable. Prove*

i. $(f(t) \cdot g(t))' = f'(t) \cdot g(t) + f(t) \cdot g'(t)$;

ii. $(f(t) \times g(t))' = f'(t) \times g(t) + f(t) \times g'(t)$;

Proof.

i. We have

$$\begin{aligned}(f(t) \cdot g(t))' &= [f_1(t)\, g_1(t) + f_2(t)\, g_2(t) + f_3(t)\, g_3(t)]' \\ &= f_1'(t)\, g_1(t) + f_1(t)\, g_1'(t) \\ &\quad + f_2'(t)\, g_2(t) + f_2(t)\, g_2'(t) \\ &\quad + f_3'(t)\, g_3(t) + f_3(t)\, g_3'(t) \\ &= f'(t) \cdot g(t) + f(t) \cdot g'(t). \end{aligned} \tag{1}$$

ii. We have

$$\begin{aligned}(f(t) \times g(t))' &= \begin{pmatrix} f_2(t)\, g_3(t) - f_3(t)\, g_2(t) \\ f_3(t)\, g_1(t) - f_1(t)\, g_3(t) \\ f_1(t)\, g_2(t) - f_2(t)\, g_1(t) \end{pmatrix}' \\ &= \begin{pmatrix} f_2'(t)\, g_3(t) + f_2(t)\, g_3'(t) - f_3'(t)\, g_2(t) - f_3(t)\, g_2'(t) \\ f_3'(t)\, g_1(t) + f_3(t)\, g_1'(t) - f_1'(t)\, g_3(t) - f_1(t)\, g_3'(t) \\ f_1'(t)\, g_2(t) + f_1(t)\, g_2'(t) - f_2'(t)\, g_1(t) - f_2(t)\, g_1'(t) \end{pmatrix} \\ &= \begin{pmatrix} f_2'(t)\, g_3(t) - f_3'(t)\, g_2(t) \\ f_3'(t)\, g_1(t) - f_1'(t)\, g_3(t) \\ f_1'(t)\, g_2(t) - f_2'(t)\, g_1(t) \end{pmatrix} + \begin{pmatrix} f_2(t)\, g_3'(t) - f_3(t)\, g_2'(t) \\ f_3(t)\, g_1'(t) - f_1(t)\, g_3'(t) \\ f_1(t)\, g_2'(t) - f_2(t)\, g_1'(t) \end{pmatrix} \\ &= f'(t) \times g(t) + f(t) \times g'(t). \end{aligned}$$

□

QUESTION 2. (5 PTS) *Let* $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$ *be defined as* $f(x) := (A\,x)\cdot x$*, where* $A = \left(\begin{smallmatrix} 4 & 5 \\ 6 & 7 \end{smallmatrix}\right)$*. Calculate the Taylor expansion of* f *to the second order at* $(1,1)$*.*

Solution. We have

$$\begin{aligned} f(x_1,x_2) &= \left[A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 4\,x_1+5\,x_2 \\ 6\,x_1+7\,x_2 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= 4\,x_1^2+11\,x_1\,x_2+7\,x_2^2. \end{aligned} \tag{2}$$

Now we calculate

$$\begin{aligned} f(1,1) &= 22 \\ \frac{\partial f}{\partial x_1}(1,1) &= 19 \\ \frac{\partial f}{\partial x_2}(1,1) &= 25 \\ \frac{\partial^2 f}{\partial x_1^2}(1,1) &= 8 \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(1,1) &= 11 \\ \frac{\partial^2 f}{\partial x_2^2}(1,1) &= 14 \end{aligned}$$

Therefore the Taylor expansion to order two is

$$22+19\,(x_1-1)+25\,(x_2-1)+\frac{1}{2}8\,(x_1-1)^2+11\,(x_1-1)\,(x_2-1)+\frac{1}{2}14\,(x_2-1)^2+R \tag{3}$$

which also equals

$$22+\begin{pmatrix} 19 \\ 25 \end{pmatrix}\cdot\begin{pmatrix} x_1-1 \\ x_2-1 \end{pmatrix}+\frac{1}{2}\left[\begin{pmatrix} 8 & 11 \\ 11 & 14 \end{pmatrix}\begin{pmatrix} x_1-1 \\ x_2-1 \end{pmatrix}\right]\cdot\begin{pmatrix} x_1-1 \\ x_2-1 \end{pmatrix}+R. \tag{4}$$

Note that for this particular function f we actually have $R=0$.

QUESTION 3. (5 PTS) *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be differentiable and non-negative with $f(0)=1$. Further assume*

$$f'(t)\leqslant t\,f(t) \tag{5}$$

for all $t\geqslant 0$. Prove that

$$f(t)\leqslant \exp(t^2/2) \tag{6}$$

for all $t\geqslant 0$.

Proof. Multiply both sides by $e^{-t^2/2}$ and move the right hand side to the left hand side, we have

$$[e^{-t^2/2}\,f]'=e^{-t^2/2}\,f'-t\,e^{-t^2/2}\,f\leqslant 0. \tag{7}$$

Therefore

$$e^{-t^2/2}\,f(t)-e^{-0^2/2}\,f(0)\leqslant 0 \tag{8}$$

which gives

$$f(t)\leqslant f(0)\,e^{t^2/2}=e^{t^2/2}. \tag{9}$$

Thus ends the proof. □

QUESTION 4. (5 PTS) *Let* $f(t)\colon \mathbb{R} \mapsto \mathbb{R}^3$ *be nonzero and smooth*[1]. *Then* $\|f(t)\|$ *is a constant* $\Longleftrightarrow f'(t) \cdot f(t) = 0$.

Note. *To prove* $A \Longleftrightarrow B$, *you need to prove two things:*

- $\Longrightarrow$*: if statement* A *is true then statement* B *is true;*
- $\Longleftarrow$*: if statement* B *is true then statement* A *is true.*

Proof.

- $\Longrightarrow$. Assume $\|f(t)\|$ is constant. Then so is $\|f(t)\|^2 = f(t) \cdot f(t)$. Consequently we have

$$0 = [f(t) \cdot f(t)]' = 2\, f'(t) \cdot f(t). \tag{10}$$

- $\Longleftarrow$. Since

$$0 = f'(t) \cdot f(t) = \frac{1}{2}\, [f(t) \cdot f(t)]' \tag{11}$$

we have $\|f(t)\|^2$ is a constant. Thus so is $\|f(t)\|$. □

1. In 348 "smooth" means there is no need to prove differentiability or integrability, no mater how many derivatives or integrals are being taken.

The following are more abstract or technical questions. They carry bonus points.

QUESTION 5. (**BONUS, 5 PTS**) *Let $f(t): \mathbb{R} \mapsto \mathbb{R}^3$ be nonzero and smooth. Then*

i. (2 PTS) *$f(t)$ has fixed direction $\Longleftrightarrow$ $f(t) \times f'(t) = 0$;*

ii. (3 PTS) *$f(t) \perp v$ for some constant vector v $\Longleftrightarrow$ $(f(t) \times f'(t)) \cdot f''(t) = 0$.*

Proof.

i.

- $\Longrightarrow$. Let v be the fixed direction. Then we have $f(t) \times v = 0$. Taking derivative we have $f'(t) \times v = 0$. Consequently $f(t) \parallel f'(t)$ and $f(t) \times f'(t) = 0$.
- $\Longleftarrow$. Assume $f(t) \times f'(t) = 0$ for all t. Let $h(t) := \frac{f(t)}{\|f(t)\|}$. Then we have $f(t) = h(t) \|f(t)\|$ which gives

$$f'(t) = h'(t) \|f(t)\| + h(t) \frac{\mathrm{d}}{\mathrm{d}t} \|f(t)\| \tag{12}$$

and

$$0 = f(t) \times f'(t) = (h(t) \times h'(t)) \|f(t)\|^2. \tag{13}$$

As $f(t)$ is nonzero, we conclude $h(t) \times h'(t) = 0$ for all t. If $h'(t) \neq 0$, then we have $h'(t) \perp h(t)$ and $\|h(t) \times h'(t)\| = \|h'(t)\| \neq 0$. Therefore $h'(t) = 0$ and consequently $h(t)$ is a constant unit vector. In other words $f(t)$ has fixed direction.

ii.

- $\Longrightarrow$. First if $f(t) \times f'(t) = 0$ then the conclusion holds automarically. In the following we assume $f(t) \times f'(t) \neq 0$. We have $f(t) \cdot v = 0$. It follows then

$$f'(t) \cdot v = [f(t) \cdot v]' = 0, \qquad f''(t) \cdot v = [f(t) \cdot v]'' = 0. \tag{14}$$

As $f(t) \times f'(t) \perp f(t), f'(t)$, there holds $f(t) \times f'(t) \parallel v$ and consequently $(f(t) \times f'(t)) \cdot f''(t) = 0$.

- $\Longleftarrow$. Let $u(t) := f(t) \times f'(t)$. We calculate

$$u'(t) = f(t) \times f''(t). \tag{15}$$

Clearly $u'(t) \perp f(t)$. On the other hand $(f''(t) \times f(t)) \cdot f'(t) = (f(t) \times f'(t)) \cdot f''(t) = 0$ which gives $u'(t) \perp f'(t)$. Now if $u(t) = 0$, then $u(t) \times u'(t) = 0$, and if $u(t) \neq 0$, we must have $u'(t) \parallel u(t)$ and again $u(t) \times u'(t) = 0$. By part i. of this problem we have $u(t) \equiv v$ is a constant vector. Thus $f(t) \cdot v = 0$ for all t and the proof ends.

□