Midterm Review II: Linear 2nd Order PDEs

We consider a general linear 2nd order PDE:

$$A(x,y)\,u_{xx}+B(x,y)\,u_{xy}+C(x,y)\,u_{yy}+D(x,y)\,u_x+E(x,y)\,u_y+F(x,y)\,u=G(x,y). \tag{1}$$

There are two types of problems we may be asked to solve:

1. Reduce the equation to its canonical form, that is one of the following three:

$$u_{\xi\eta}=\cdots,\quad u_{\eta\eta}=\cdots,\quad u_{\xi\xi}+u_{\eta\eta}=\cdots. \tag{2}$$

   One thing to keep in mind is that the variables in canonical forms are all real (otherwise there is no difference between the hyperbolic case and the elliptic case anymore).

2. Find the general solutions.

The procedures of solution are slightly different.

1. Reduction to canonical forms.

1. Determine its type (which may vary at different points):

$$B^2-4\,A\,C\begin{cases}>0 & \text{hyperbolic} \quad (\Longrightarrow\ u_{\xi\eta}=\cdots)\\ =0 & \text{parabolic} \quad (\Longrightarrow\ u_{\eta\eta}=\cdots)\\ <0 & \text{elliptic} \quad (\Longrightarrow\ u_{\alpha\alpha}+u_{\beta\beta}=\cdots)\end{cases}. \tag{3}$$

2. Write down the characteristics equation

$$A(x,y)\,(\mathrm{d}y)^2-B(x,y)\,(\mathrm{d}x)\,(\mathrm{d}y)+C(x,y)\,(\mathrm{d}x)^2=0. \tag{4}$$

   Factorize:

$$\begin{array}{lll} \text{hyperbolic} \Longrightarrow & (m(x,y)\,\mathrm{d}y+n(x,y)\,\mathrm{d}x)\,(m'(x,y)\,\mathrm{d}y+n'(x,y)\,\mathrm{d}x)=0 & \Longrightarrow\ \mathrm{d}\xi(x,y)=\mathrm{d}\eta(x,y)=0\\ \text{parabolic} \Longrightarrow & (m(x,y)\,\mathrm{d}y+n(x,y)\,\mathrm{d}x)^2=0 & \Longrightarrow\ \mathrm{d}\xi(x,y)=0\\ \text{elliptic} \Longrightarrow & (m(x,y)\,\mathrm{d}y+n(x,y)\,\mathrm{d}x)\,(m'(x,y)\,\mathrm{d}y+n'(x,y)\,\mathrm{d}x)=0 & \Longrightarrow\ \mathrm{d}\xi(x,y)=\mathrm{d}\eta(x,y)=0 \end{array}$$

   Note that in the elliptic case, ξ,η are complex.

3. Determine new varaibles

hyperbolic	just use ξ,η
parabolic	use ξ, and pick any η such that $\det\begin{pmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{pmatrix}\neq 0$ First choices are $\eta=x$ or y
elliptic	use $\alpha=\frac{\xi+\eta}{2}$, $\beta=\frac{\xi-\eta}{2\,i}$.

4. Do the change of variables. Replace (use α,β instead of ξ,η in the elliptic case)

$$\begin{array}{lll} u & \text{by} & u\\ u_x & \text{by} & u_\xi\,\xi_x+u_\eta\,\eta_x\\ u_y & \text{by} & u_\xi\,\xi_y+u_\eta\,\eta_y\\ u_{xx} & \text{by} & u_{\xi\xi}\,\xi_x^2+2\,u_{\xi\eta}\,\xi_x\,\eta_x+u_{\eta\eta}\,\eta_x^2+u_\xi\,\xi_{xx}+u_\eta\,\eta_{xx}\\ u_{xy} & \text{by} & u_{\xi\xi}\,\xi_x\,\xi_y+u_{\xi\eta}\,(\xi_x\,\eta_y+\xi_y\,\eta_x)+u_{\eta\eta}\,\eta_x\,\eta_y+u_\xi\,\xi_{xy}+u_\eta\,\eta_{xy}\\ u_{yy} & \text{by} & u_{\xi\xi}\,\xi_y^2+2\,u_{\xi\eta}\,\xi_y\,\eta_y+u_{\eta\eta}\,\eta_y^2+u_\xi\,\xi_{yy}+u_\eta\,\eta_{yy} \end{array}$$

   in the equation. Finally invert $\xi=\xi(x,y)$, $\eta=\eta(x,y)$ to represent x,y by ξ,η, thus obtain a new equation with ξ,η (α,β in the elliptic case). This new equation is the desired canonical form.

2. Finding general solutions.

For hyperbolic and parabolic equations, reduction to canonical form is the first step of finding general solutions. On the other hand, for elliptic equations, there is no need to reduce to canonical form first. What we do is the following.

1. Determine the complex new variables ξ,η.
2. Do the change of variables, reduce the equation to the form

$$u_{\xi\eta}=\cdots \tag{5}$$

3. Examples.

Example 1. (§4.6, 7) Reduce the Tricomi equation

$$u_{xx} + x\,u_{yy} = 0 \tag{6}$$

to the canonical form.

Solution.

1. We have $A = 1, B = 0, C = x$, and $B^2 - 4\,A\,C = -4\,x$. Thus the equation is hyperbolic when $x < 0$ and elliptic when $x > 0$.

2. The characteristics equation reads

$$(\mathrm{d}y)^2 + x\,(\mathrm{d}x)^2 = 0 \tag{7}$$

which gives

$$\left(\mathrm{d}y - \sqrt{-x}\,\mathrm{d}x\right)\left(\mathrm{d}y + \sqrt{-x}\,\mathrm{d}x\right) = 0 \tag{8}$$

when $x < 0$ and

$$\left(\mathrm{d}y - i\,\sqrt{x}\,\mathrm{d}x\right)\left(\mathrm{d}y + i\,\sqrt{x}\,\mathrm{d}x\right) = 0 \tag{9}$$

when $x > 0$. Correspondingly, we have

- $\xi(x, y) = y + \frac{2}{3}\,(-x)^{3/2}$, $\eta = y - \frac{2}{3}\,(-x)^{3/2}$ when $x < 0$, and
- $\xi(x, y) = y - i\,\frac{2}{3}\,x^{3/2}$, $\eta = y + i\,\frac{2}{3}\,x^{3/2}$ when $x > 0$.

3. & 4.

- Hyperbolic case ($x < 0$): We use ξ, η as the new variables, we compute

$$\xi_x = -\sqrt{-x},\ \ \xi_y = 1,\ \ \xi_{xx} = \frac{1}{2\,\sqrt{-x}},\ \ \xi_{xy} = 0,\ \ \xi_{yy} = 0, \tag{10}$$

$$\eta_x = \sqrt{-x},\ \ \eta_y = 1,\ \ \eta_{xx} = -\frac{1}{2\,\sqrt{-x}},\ \ \eta_{xy} = 0,\ \ \eta_{yy} = 0. \tag{11}$$

Thus

$$u_{xx} = u_{\xi\xi}\,(-x) + 2\,x\,u_{\xi\eta} + u_{\eta\eta}\,(-x) + u_\xi\,\frac{1}{2\,\sqrt{-x}} + u_\eta\left(-\frac{1}{2\,\sqrt{-x}}\right), \tag{12}$$

$$u_{yy} = u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta}. \tag{13}$$

We have

$$\begin{aligned} u_{xx} + x\,u_{yy} &= -x\,u_{\xi\xi} + 2\,x\,u_{\xi\eta} - x\,u_{\eta\eta} + u_\xi\,\frac{1}{2\,\sqrt{-x}} + u_\eta\left(-\frac{1}{2\,\sqrt{-x}}\right) \\ &\quad + x\,u_{\xi\xi} + 2\,x\,u_{\xi\eta} + x\,u_{\eta\eta} \\ &= 4\,x\,u_{\xi\eta} + \frac{1}{2\,\sqrt{-x}}\,(u_\xi - u_\eta). \end{aligned} \tag{14}$$

Thus

$$u_{\xi\eta} = \frac{1}{8\,(-x)^{3/2}}\,(u_\xi - u_\eta). \tag{15}$$

Finally solving

$$(-x)^{3/2} = \frac{3}{4}\,(\xi - \eta) \tag{16}$$

gives

$$u_{\xi\eta} = \frac{1}{6\,(\xi - \eta)}\,(u_\xi - u_\eta). \tag{17}$$

- Elliptic case ($x > 0$): We introduce new variables

$$\alpha = \frac{\xi + \eta}{2} = y, \quad \beta = \frac{\xi - \eta}{2\,i} = -\frac{2}{3}\,x^{3/2}. \tag{18}$$

(Note that if we are trying to find general solutions, we would use ξ, η). We compute

$$\alpha_x = 0, \;\; \alpha_y = 1, \;\; \alpha_{xx} = \alpha_{xy} = \alpha_{yy} = 0, \tag{19}$$

$$\beta_x = -x^{1/2}, \;\; \beta_y = 0, \;\; \beta_{xx} = -\frac{1}{2}x^{-1/2}, \;\; \beta_{xy} = \beta_{yy} = 0. \tag{20}$$

Consequently

$$u_{xx} = x\,u_{\beta\beta} + u_\beta\left(-\frac{1}{2}x^{-1/2}\right) \tag{21}$$

$$u_{yy} = u_{\alpha\alpha} \tag{22}$$

Thus

$$u_{xx} + x\,u_{yy} = x\,u_{\beta\beta} + u_\beta\left(-\frac{1}{2}x^{-1/2}\right) + x\,u_{\alpha\alpha} \tag{23}$$

and the equation

$$u_{xx} + x\,u_{yy} = 0 \tag{24}$$

becomes

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{2}x^{-3/2}u_\beta. \tag{25}$$

Finally

$$\alpha = y, \quad \beta = -\frac{2}{3}x^{3/2} \Longrightarrow \frac{1}{2}x^{-3/2} = -\frac{1}{3\beta} \tag{26}$$

which leads to

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\beta}{3\beta}. \tag{27}$$

(Note: In the book the above equation is further simplified. But that's out of our scope here.)