Midterm Review I: Method of Characteristics

Recall that we try to find general solutions of the following quasi-linear first order PDE:

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u). \tag{1}$$

1. The "algorithm".

We will present the method of characteristics in a slightly different manner which is more "algorithm-like".

The initial set-up is the following "chain" of equalities:

$$\frac{\mathrm{d}x}{a(x, y, u)} = \frac{\mathrm{d}y}{b(x, y, u)} = \frac{\mathrm{d}u}{c(x, y, u)}. \tag{2}$$

This is an "equality-chain" of formal ratios.

In the "main part" of method of characteristics, we add more and more new terms (ratios) to this chain by applying the following property of equivalent ratios repeatedly (each application adds exactly one new ratio to the chain):

$$\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n} \Longrightarrow \frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n} = \frac{c_1\, a_1 + c_2\, a_2 + \cdots + c_n\, a_n}{c_1\, b_1 + c_2\, b_2 + \cdots + c_n\, b_n}, \tag{3}$$

where each c_i can be either a number or a function of x, y, u.

The "algorithm" terminates when the "chain" contains two (three when we have three variables x, y, z) ratios of the form[1] $\frac{\mathrm{d}\phi(x, y, u)}{0}$ and $\frac{\mathrm{d}\psi(x, y, u)}{0}$ and furthermore $\phi(x,\, y,\, u)$ and $\psi(x,\, y,\, u)$ are "independent" (meaning there is no function h such that $h(\phi) = \psi$).

As soon as ϕ and ψ are obtained, the general solution can be written as

$$F(\phi, \psi) = 0 \tag{4}$$

where F is any function. Fix one F, we obtain one solution of the equation. Note that the correspondence is not one-to-one, in particular different F may give the same solution.

Remark 1. If we just form new ratios randomly by taking random $c_1, \ldots, c_n$, then almost certainly our new ratios will be useless. Although there is no "universal rule" to follow to obtain "useful" new ratios, there are indeed some guidelines. In short, we try to find a set of multipliers $c_1, \ldots, c_n$ which will make the denominator 0. In particular one can aim at obtaining the following "intermediate" ratios:

$$\frac{\mathrm{d}f}{f} = \pm \frac{\mathrm{d}g}{g}; \quad \frac{\mathrm{d}f}{g} = \pm \frac{\mathrm{d}g}{f} \tag{5}$$

because, for example,

$$\frac{\mathrm{d}f}{f} = \frac{\mathrm{d}g}{g} \Longrightarrow \frac{\left(f^{-1}\right) \mathrm{d}f - \left(g^{-1}\right) \mathrm{d}g}{\left(f^{-1}\right) f - \left(g^{-1}\right) g} \Longrightarrow \frac{\mathrm{d}(\ln f - \ln g)}{0}; \tag{6}$$

$$\frac{\mathrm{d}f}{g} = -\frac{\mathrm{d}g}{f} \Longrightarrow \frac{f\, \mathrm{d}f - g\,(-\mathrm{d}g)}{f g - g f} \Longrightarrow \frac{\mathrm{d}\left[\frac{1}{2}\left(f^2 + g^2\right)\right]}{0}. \tag{7}$$

In the first one we have taken $c_1 = f^{-1}, c_2 = -g^{-1}$, in the second one $c_1 = f, c_2 = -g$.

2. Examples.

Example 2. (§2.8, 8 e)) Solve

$$x\left(y^2 - z^2\right) u_x + y\left(z^2 - x^2\right) u_y + z\left(x^2 - y^2\right) u_z = 0. \tag{8}$$

(There is a typo in the book)

Solution. We "initialize" the "chain":

$$\frac{\mathrm{d}x}{x\left(y^2 - z^2\right)} = \frac{\mathrm{d}y}{y\left(z^2 - x^2\right)} = \frac{\mathrm{d}z}{z\left(x^2 - y^2\right)} = \frac{\mathrm{d}u}{0}. \tag{9}$$

1. We see that it's OK to be "divided by 0" in the method. The reason to this tolerance will be explained in §3.

We see that there is one ratio with denominator 0, we need to find the other two.

Recalling that our main tool is

$$\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n} \Longrightarrow \frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n} = \frac{c_1\, a_1 + c_2\, a_2 + \cdots + c_n\, a_n}{c_1\, b_1 + c_2\, b_2 + \cdots + c_n\, b_n}, \tag{10}$$

We start by trying to find a set of multipliers which can yield a 0 denominator.

Observing the special symmetry of the denominators, we have

$$\left(x^{-1}\right)\left[x\left(y^2 - z^2\right)\right] + \left(y^{-1}\right)\left[y\left(z^2 - x^2\right)\right] + \left(z^{-1}\right)\left[z\left(x^2 - y^2\right)\right] = 0. \tag{11}$$

In other words, we can have a new ratio

$$\frac{x^{-1}\,\mathrm{d}x + y^{-1}\,\mathrm{d}y + z^{-1}\,\mathrm{d}z}{0}. \tag{12}$$

The question is whether the nominator can be written as $\mathrm{d}\phi$. We are lucky enough that

$$\frac{x^{-1}\,\mathrm{d}x + y^{-1}\,\mathrm{d}y + z^{-1}\,\mathrm{d}z}{0} = \frac{\mathrm{d}(\ln x + \ln y + \ln z)}{0} = \frac{\mathrm{d}[\ln(x\,y\,z)]}{0}. \tag{13}$$

Note that as $\frac{\mathrm{d}f}{0} \Longrightarrow \frac{\mathrm{d}g(f)}{0}$ for any function g, we have

$$\frac{\mathrm{d}(x\,y\,z)}{0} \tag{14}$$

in the chain.

The thrid ratio can finally be obtained by observing

$$x\left[x\left(y^2 - z^2\right)\right] + y\left[y\left(z^2 - x^2\right)\right] + z\left[z\left(x^2 - y^2\right)\right] = 0 \tag{15}$$

which gives

$$\frac{\mathrm{d}\left(x^2 + y^2 + z^2\right)}{0}. \tag{16}$$

Thus we have all three “useful” ratios

$$\frac{\mathrm{d}u}{0}, \quad \frac{\mathrm{d}(x\,y\,z)}{0}, \quad \frac{\mathrm{d}\left(x^2 + y^2 + z^2\right)}{0} \tag{17}$$

which leads to the general solution

$$F\left(u, x\,y\,z, x^2 + y^2 + z^2\right) = 0 \Longrightarrow u = f\left(x\,y\,z, x^2 + y^2 + z^2\right). \tag{18}$$

Note that there can be many failed attempts, for example, we observe that $\frac{\mathrm{d}z}{z\,(x^2 - y^2)}$ can be written as $\frac{\mathrm{d}(\ln z)}{x^2 - y^2}$ where the nominator only contains z and the denominator only x, y, so we try to combine the first two ratios to obtain $\frac{\mathrm{d}(x^2 - y^2)}{\text{function of } z}$. But we do not succeed.

3. A bit explanation.

Now we try to give some reason to our tolerance of ratios like $\frac{\mathrm{d}\phi}{0}$. Recall our equation

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u). \tag{19}$$

Let $u = u(x, y)$ be a solution. Now consider a curve in the x-y plane:

$$x = x(s), \quad y = y(s) \tag{20}$$

We have

$$u = u(x(s), y(s)). \tag{21}$$

Taking s derivative we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}s} = x'(s)\, u_x + y'(s)\, u_y. \tag{22}$$

Comparing with the equation we see that when $x'(s) = a(x(s), y(s), u(s))$, $y'(s) = b(x(s), y(s), u(s))$ we must have

$$u'(s) = c(x(s), y(s), u(s)). \tag{23}$$

This can be re-written as

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{c} = \mathrm{d}s \tag{24}$$

along this particular curve $(x(s), y(s), u(s))$.

Therefore $\frac{\mathrm{d}\phi(x,\, y,\, u)}{0}$ is actually a "short-hand" of

$$\frac{\mathrm{d}\phi(x(s), y(s), u(s))}{\mathrm{d}s} = 0 \tag{25}$$

which makes perfect sense, and means that the curve $(x(s),\, y(s),\, u(s))$ is the intersection of the solution $u - u(x, y) = 0$ and some level set of ϕ: $\phi(x, y, u) = c$.

Now if we have another ψ, independent of ϕ (meaning: any level set of ψ intersects any level set of ϕ along a curve), such that $\frac{\mathrm{d}\psi}{0}$ appears in the chain, then the curve $(x(s),\, y(s),\, u(s))$ is also the intersection of the solution $u - u(x, y) = 0$ and some level set of ϕ: $\psi(x, y, u) = c'$.

Now it is clear that the curve $(x(s),\, y(s),\, u(s))$ has to coincide with the intersection of $\phi(x,\, y,\, u) = c$ and $\psi(x, y, u) = c'$, in other words, the curve is contained in

$$F(\phi, \psi) = 0 \tag{26}$$

for some F.