Green's Function

In most of our lectures we only deal with initial and boundary value problems of homogeneous equation. How about nonhomogeneous equations whose RHS are not 0? For example, consider the Laplace equation

$$u_{xx} + u_{yy} = f(x, y) \qquad x^2 + y^2 < 1 \tag{1}$$

$$u(x, y) = 0 \qquad x^2 + y^2 = 1. \tag{2}$$

One approach is to follow the idea of separation of variables. More specifically, we rewrite the problem in polar coordinates

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = f(r, \theta) \qquad r < 1 \tag{3}$$

$$u(1, \theta) = 0 \tag{4}$$

Then, motivated by the result of the corresponding homogeneous problem, we write the solution as

$$u(r, \theta) = \sum_{n=0}^{\infty} [u_{n,1}(r) \cos(n\,\theta) + u_{n,2}(r) \sin(n\,\theta)]. \tag{5}$$

To find out $u_{n,1}(r)$ and $u_{n,2}(r)$, we expand

$$f(r, \theta) = \sum_{n=0}^{\infty} [f_{n,1}(r) \cos(n\,\theta) + f_{n,2}(r) \sin(n\,\theta)]. \tag{6}$$

Substituting these into the equation and equating the terms involving the same trignometric functions, we have

$$u_{n,1}''(r) + \frac{1}{r} u_{n,1}'(r) + \frac{1}{r^2} \left(- n^2 u_{n,1}(r) \right) = f_{n,1}(r) \tag{7}$$

$$u_{n,1}(1) = 0 \tag{8}$$

and

$$u_{n,2}''(r) + \frac{1}{r} u_{n,2}'(r) + \frac{1}{r^2} \left(- n^2 u_{n,2}(r) \right) = f_{n,2}(r) \tag{9}$$

$$u_{n,2}(1) = 0. \tag{10}$$

We need to solve these equations to obtain an infinite sum formula of the solution.

Now there are two problems with this approach.

1. How do we solve the ODEs for $u_{n,1}$ and $u_{n,2}$?
2. (More importantly) How do we figure out the real-space dependence of u on the data f? For example, if we know the shape of the function $f(x, y)$, can we have any idea of the shape of the function $u(x, y)$?

The answers the the above questions lie in the theory of Green's functions. A Green's function for a particular differential operator over a particular domain involving with n variables $x_1, \ldots, x_n$ is a function of $2\,n$ variables $x_1, \ldots, x_n; \xi_1, \ldots, \xi_n$, denoted $G(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$, which can produce the solution from the data: the RHS and the boundary values.

For example, for the above example, the Green's function corresponding to the differential operator $\partial_{xx} + \partial_{yy}$ over the domain $x^2 + y^2 < 1$ is a function $G(x, y; \xi, \eta)$ such that for any RHS $f(x, y)$, the solution is given, in essence, by the following formula

$$u(x, y) = \int_{\xi^2 + \eta^2 < 1} G(x, y; \xi, \eta)\, f(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11}$$

We see that as soon as we know the Green's function of a problem, we have a universal formula fo its solutions.

Remark 1. The precise relation between the Green's function and the problem is the correspondence

Green's function $\longleftrightarrow$ (Differential operator, Domain, Type of Boundary Conditions)

Now how do we figure out the Green's function of a given problem

$$L_x u = f? \tag{12}$$

If we take the formula

$$u = \int G(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)\, f(\xi_1, \ldots, \xi_n)\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n \tag{13}$$

and put it into the equation, we have

$$\int (L_x G(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n))\, f(\xi_1, \ldots, \xi_n)\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n = f(x_1, \ldots, x_n). \tag{14}$$

To figure out how this helps in finding G, we need knowledge of the Dirac delta function.

1. The Dirac delta function.

The Dirac delta "function" is a non-traditional function which can only be defined by its action on continuous functions:

$$\int_{\mathbb{R}^n} \delta(x_1, x_2, \ldots, x_n)\, f(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = f(0, 0, \ldots, 0). \tag{15}$$

Remark 2. Interested readers can try to show that no continuous function can have the above property.

In particular, the 1D/2D delta functions are defined by

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, \mathrm{d}x = f(0); \qquad \iint \delta(x, y)\, f(x, y)\, \mathrm{d}x\, \mathrm{d}y = f(0, 0). \tag{16}$$

To make effective use of the delta function, we develop the following properties. In the following we only show cases up to 2 dimension, higher dimension cases are similar.

1. Relation between 1D delta function and higher dimensional delta functions.

   Since

   $$\iint \delta(x)\, \delta(y)\, f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_{-\infty}^{\infty} \delta(x) \left[\int_{-\infty}^{\infty} \delta(y)\, f(x, y)\, \mathrm{d}y \right] \mathrm{d}x = f(0, 0). \tag{17}$$

   Therefore

   $$\delta(x, y) = \delta(x)\, \delta(y). \tag{18}$$

2. Translated delta functions (which is a special case of the following "change of variables" property). It is easy to show

   $$\int \delta(x - \xi)\, f(x)\, \mathrm{d}x = f(\xi), \qquad \iint \delta(x - \xi, y - \eta)\, f(x, y)\, \mathrm{d}x\, \mathrm{d}y = f(\xi, \eta). \tag{19}$$

3. Behavior of the delta function under general change of variables.

   We first consider the 1D case. Consider the change of variable $\alpha = \alpha(x)$ or equivalently $x = x(\alpha)$. Consider $\delta(x - \xi)$. We try to find its representation in the new variable. In other words, we try to compute

   $$\int \delta(x - \xi)\, f(\alpha)\, \mathrm{d}\alpha. \tag{20}$$

   Now as $\alpha = \alpha(x)$, we have

   $$\begin{aligned} \int \delta(x - \xi)\, f(\alpha)\, \mathrm{d}\alpha &= \int \delta(x - \xi)\, f(\alpha(x))\, \alpha'(x)\, \mathrm{d}x \\ &= f(\alpha(\xi))\, \alpha'(\xi) \\ &= \int \alpha'(\xi)\, \delta(\alpha - \alpha(\xi))\, f(\alpha)\, \mathrm{d}\alpha \end{aligned} \tag{21}$$

   which yields

   $$\delta(x - \xi) = \alpha'(\xi)\, \delta(\alpha - \alpha(\xi)). \tag{22}$$

Similarly, in 2D, under the change of variables

$$\alpha = \alpha(x, y), \qquad \beta = \beta(x, y) \tag{23}$$

we have

$$\begin{aligned} \iint \delta(x-\xi, y-\eta)\, f(\alpha, \beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta &= \iint \delta(x-\xi, y-\eta)\, f(\alpha(x, y), \beta(x, y))\, |J(x, y)|\, \mathrm{d}x\, \mathrm{d}y \\ &= |J(\xi, \eta)|\, f(\alpha(\xi, \eta), \beta(\xi, \eta)) \end{aligned} \tag{24}$$

which gives

$$\delta(x-\xi, y-\eta) = |J(\xi, \eta)|\, \delta(\alpha - \alpha(\xi, \eta), \beta - \beta(\xi, \eta)) \tag{25}$$

where

$$J(x, y) = \det \begin{pmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{pmatrix} \tag{26}$$

is the Jacobian of the change of variables.

Example 3. Find the representation of the 2D delta function in polar coordinates.

Solution. We would like to obtain the formula for $\delta(x-\xi, y-\eta)$ in the polar coordinates (r, θ).

The change of variables is given by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{27}$$

Differentiating, we obtain

$$r_x = \frac{x}{r}, \qquad r_y = \frac{y}{r}, \qquad \theta_x = -\frac{y}{r^2}, \qquad \theta_y = \frac{x}{r^2} \tag{28}$$

and therefore

$$J(x, y) = \det \begin{pmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{pmatrix} = \frac{1}{r}. \tag{29}$$

Now let ρ, γ be such that

$$\xi = \rho\cos\gamma, \qquad \eta = \rho\sin\gamma, \tag{30}$$

we finally have

$$\delta(x-\xi, y-\eta) = \frac{1}{\rho}\, \delta(r-\rho, \theta-\gamma). \tag{31}$$

We note that the above formula breaks down when $\rho = 0$, because in that case γ can be any angle. Therefore this case needs to be treated separately.

To find out the formula for $\delta(x, y)$, we return to the definition. Denote the (unknown) formula for $\delta(x, y)$ in polar coordinates by $\Delta(r, \theta)$.

$$\begin{aligned} f(0,0) &= \iint \delta(x, y)\, f(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^{2\pi} \int \Delta(r, \theta)\, f(r, \theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta \end{aligned} \tag{32}$$

Now notice that, for any function $f_\gamma(r, \theta) = f(r, \theta + \gamma)$ we have (in Cartesian coordinates) $f(0, 0) = f_\gamma(0, 0)$. Therefore

$$\int_0^{2\pi} \int \Delta(r, \theta)\, f(r, \theta + \gamma)\, r\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int \Delta(r, \theta)\, f(r, \theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{33}$$

This implies that $\Delta(r, \theta)$ is independent of θ, from now on we denote it as $\Delta(r)$.

Now we have

$$f(0, 0) = \int_0^{2\pi} \left[\int \Delta(r)\, f(r, \theta)\, r\, \mathrm{d}r \right] \mathrm{d}\theta. \tag{34}$$

It is clear that we should take

$$\Delta(r) = \frac{1}{2\pi} \frac{\delta(r)}{r}. \tag{35}$$

2. Green's function for ODEs.

We consider the ODE system

$$
\begin{aligned}
(p(x)\;y'(x))' + q(x)\,y(x) &= f(x) \qquad a<x<b && (36)\\
a_1\,y(a) + a_2\,y'(a) &= 0 && (37)\\
b_1\,y(b) + b_2\,y'(b) &= 0. && (38)
\end{aligned}
$$

Beware the difference of our equation and (8.11.1) on page 310 of the textbook!

Recall that we would like to have

$$\int G(x;\xi)\,f(\xi)\,\mathrm{d}\xi = y(x). \tag{39}$$

As $f(\xi)$ is only given for $a<\xi<b$, the limits of the integral are naturally a and b. Therefore we need

$$\int_a^b G(x;\xi)\,f(\xi)\,\mathrm{d}\xi = y(x). \tag{40}$$

Now using the fact that

$$(p\,y')' + q\,y = f \tag{41}$$

the above becomes

$$\int_a^b G(x;\xi)\left[(p\,y')' + q\,y\right]\mathrm{d}\xi = y(x) \tag{42}$$

the $'$ in the above integral denotes derivatives in ξ.

We use integration by parts to compute

$$
\begin{aligned}
\int_a^b G(x;\xi)\left[(p\,y')' + q\,y\right]\mathrm{d}\xi &= \int_a^b G(x;\xi)\,\mathrm{d}(p\,y') + \int_a^b q\,G\,y\\
&= G(x;\xi)\,p\,y'\,|_a^b - \int_a^b p\,y'\,G'\,\mathrm{d}\xi + \int_a^b q\,G\,y\\
&= G(x;b)\,p(b)\,y'(b) - G(x;a)\,p(a)\,y'(a) - \int_a^b p\,G'\,\mathrm{d}y + \int_a^b q\,G\,y\\
&= G(x;b)\,p(b)\,y'(b) - G(x;a)\,p(a)\,y'(a) - p\,G'\,y\,|_a^b + \int_a^b \left[(p\,G')' + q\,G\right]y\\
&= G(x;b)\,p(b)\,y'(b) - G(x;a)\,p(a)\,y'(a) - p(b)\,G'(b)\,y(b) + p(a)\,G'(a)\,y(a)\\
&\quad + \int_a^b \left[(p\,G')' + q\,G\right]y\,\mathrm{d}\xi\\
&= p(b)\left[G(x;b)\,y'(b) - G'(x;b)\,y(b)\right] - p(a)\left[G(x;a)\,y'(a) - G'(x;a)\,y(a)\right]\\
&\quad + \int_a^b \left[(p\,G')' + q\,G\right]y\,\mathrm{d}\xi. && (43)
\end{aligned}
$$

Thus we see that the conditions on G should be

$$
\begin{aligned}
G(x;b)\,y'(b) - G'(x;b)\,y(b) &= 0 && (44)\\
G(x;a)\,y'(a) - G'(x;a)\,y(a) &= 0 && (45)\\
\Big(p(\xi)\,G(x;\xi)_\xi\Big)_\xi + q(\xi)\,G(x;\xi) &= \delta(x-\xi). && (46)
\end{aligned}
$$

Instead of solving this system directly, we cite the following property of a Green's function[1]

$$G(x;\xi) = G(\xi;x) \tag{48}$$

1. A non-rigorous "proof" is the following. For any Green's function $G(x;\,\xi)$ we have $L_\xi G(x;\,\xi) = \delta(x-\xi)$. In other words, if we denote $u(\xi) = G(x;\xi)$, then $L_\xi u = \delta(x-\xi)$. Now since G is the Green's function, we have

$$G(x;\xi) = u(\xi) = \int G(\xi;\,y)\,\delta(x-y)\,\mathrm{d}y = G(\xi;x). \tag{47}$$

to replace the above system by the following:

$$\begin{aligned}\big(p(x)\,G(x;\xi)_x\big)_x + q(x)\,G(x;\xi) &= \delta(x-\xi) & (49)\\ G(b;\xi)\,y'(b) - G_x(b;\xi)\,y(b) &= 0 & (50)\\ G(a;\xi)\,y'(a) - G_x(a;\xi)\,y(a) &= 0. & (51)\end{aligned}$$

Note that this implies G should satisfy the same boundary condition as y.

Now we solve the system as follows. Since $\delta(x-\xi)=0$ for all $x\neq\xi$, we can break the above equation into two equations:

$$\big[p(x)\,G(x;\xi)_x\big]_x + q(x)\,G(x;\xi) = 0, \qquad a_1\,G(a;\xi) + a_2\,G_x(a;\xi) = 0, \qquad a<x<\xi \tag{52}$$

$$\big[p(x)\,G(x;\xi)_x\big]_x + q(x)\,G(x;\xi) = 0, \qquad b_1\,G(b;\xi) + b_2\,G_x(b;\xi) = 0, \qquad \xi<x<b. \tag{53}$$

The plan now is the following. Each ODE system gives us a family of solutions with one degree of freedom. Then we would choose the parameters appropriately to obtain the correct Green's function.

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation over $a<x<b$, chosen such that

$$a_1\,y_1(a) + a_2\,y_1'(a) = 0; \qquad b_1\,y_2(b) + b_2\,y_2'(b) = 0. \tag{54}$$

Then we have

$$G(x;\xi) = \begin{cases} c_1(\xi)\,y_1(x) & x<\xi \\ c_2(\xi)\,y_2(x) & x>\xi \end{cases}. \tag{55}$$

Note that for any $c_1(\xi), c_2(\xi)$, the corresponding $G(x;\xi)$ satisfies

$$\big[p(x)\,G(x;\xi)_x\big]_x + q(x)\,G(x;\xi) = 0 \qquad x\neq\xi. \tag{56}$$

Our last task is to figure out the appropriate $c_1(\xi)$ and $c_2(\xi)$ so that

$$\big[p(x)\,G(x;\xi)_x\big]_x + q(x)\,G(x;\xi) = \delta(x-\xi). \tag{57}$$

To do this we recall the definition of the δ function

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,f(x)\,\mathrm{d}x = f(\xi) \tag{58}$$

for any function f which is smooth and satisfying $f(\xi\pm\varepsilon) = f'(\xi\pm\varepsilon) = 0$. Thus the requirement on G can be revealed by

$$\begin{aligned} f(\xi) &= \int_{\xi-\varepsilon}^{\xi+\varepsilon} \Big\{\big[p(x)\,G(x;\xi)_x\big]_x + q(x)\,G(x;\xi)\Big\}\,f(x)\,\mathrm{d}x \\ &= \int_{\xi-\varepsilon}^{\xi+\varepsilon} \big[p(x)\,G(x;\xi)_x\big]_x\,f(x)\,\mathrm{d}x + \int_{\xi-\varepsilon}^{\xi+\varepsilon} q(x)\,G(x;\xi)\,f(x)\,\mathrm{d}x. \end{aligned} \tag{59}$$

Assuming that G is bounded, the second term vanishes as $\varepsilon\searrow 0$. Therefore we have

$$f(\xi) = \lim_{\varepsilon\searrow 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon} \big[p(x)\,G(x;\xi)_x\big]_x\,f(x)\,\mathrm{d}x. \tag{60}$$

Or equivalently

$$p(\xi+)\,G_x(\xi+;\xi) - p(\xi-)\,G_x(\xi-;\xi) = 1. \tag{61}$$

Recalling

$$G(x;\xi) = \begin{cases} c_1(\xi)\,y_1(x) & x<\xi \\ c_2(\xi)\,y_2(x) & x>\xi \end{cases}. \tag{62}$$

we have

$$G(\xi+;\xi) = G(\xi-;\xi) \implies c_2(\xi)\,y_2(\xi) = c_1(\xi)\,y_1(\xi), \tag{63}$$

$$p(\xi+)\,G_x(\xi+;\xi) - p(\xi-)\,G_x(\xi-;\xi) = 1 \implies p(\xi)\,c_2(\xi)\,y_2'(\xi) - p(\xi)\,c_1(\xi)\,y_1'(\xi) = 1. \tag{64}$$

Solving these two equations, we have

$$c_1(\xi) = \frac{y_2(\xi)}{p(\xi)\,[y_1(\xi)\,y_2'(\xi) - y_2(\xi)\,y_1'(\xi)]}, \quad c_2 = \frac{y_1(\xi)}{p(\xi)\,[y_1(\xi)\,y_2'(\xi) - y_2(\xi)\,y_1'(\xi)]}. \tag{65}$$

To further simpify, we notice that from

$$\left(p\,y_1'\right)' + q\,y_1 = 0 \quad \text{and} \quad \left(p\,y_2'\right)' + q\,y_2 = 0 \tag{66}$$

we conclude that

$$\left[p\left(y_1\,y_2' - y_2\,y_1'\right)\right]' = 0. \tag{67}$$

Therefore

$$p(\xi)\left[y_1(\xi)\,y_2'(\xi) - y_2(\xi)\,y_1'(\xi)\right] = C \tag{68}$$

is a constant.

Summarizing, we have

$$G(x;\xi) = \begin{cases} \dfrac{y_2(\xi)\,y_1(x)}{C} & x < \xi \\ \dfrac{y_1(\xi)\,y_2(x)}{C} & x > \xi \end{cases}. \tag{69}$$

Example 4. (§8.14 11 b) Find the Green's function for the following problem

$$\left(1 - x^2\right)y'' - 2\,x\,y' = 0, \qquad y(0) = 0, \quad y'(1) = 0. \tag{70}$$

Solution. We notice that

$$\left(1 - x^2\right)y'' - 2\,x\,y' = \left[\left(1 - x^2\right)y'\right]'. \tag{71}$$

Therefore

$$\left(1 - x^2\right)y' = c \implies y' = \frac{c}{1 - x^2}. \tag{72}$$

Taking $c = 0$ we obtain $y = \text{constant}$; For $c \neq 0$, we have

$$y' = \frac{c}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) \Longrightarrow y = -\frac{c}{2}\ln\left(\frac{1+x}{1-x}\right) + \text{constant}. \tag{73}$$

Thus the general solution is

$$y = c_1 \ln\left(\frac{1+x}{1-x}\right) + c_2. \tag{74}$$

Requiring $y(0) = 0$ and $y'(1) = 0$ we obtain

$$y_1(x) = \ln\left(\frac{1+x}{1-x}\right), \qquad y_2(x) = 1. \tag{75}$$

As the equation is $\left[\left(1 - x^2\right)y'\right]' = 0$ we have $p(x) = \left(1 - x^2\right)$.

Thus

$$C = p(\xi)\left[y_1(\xi)\,y_2'(\xi) - y_2(\xi)\,y_1'(\xi)\right] = -2. \tag{76}$$

Therefore

$$G(x;\xi) = \begin{cases} -\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) & x < \xi \\ -\dfrac{1}{2}\ln\left(\dfrac{1+\xi}{1-\xi}\right) & x > \xi \end{cases}. \tag{77}$$

Example 5. (§8.14 12 a) Determine the solution of the following boundary-value problem.

$$y'' + y = 1, \qquad y(0) = 0, \quad y(1) = 0. \tag{78}$$

Solution. We solve the problem using Green's function.

First we calculate Green's function. The general solutions for the homogeneous problem

$$y'' + y = 0 \tag{79}$$

is

$$y = A \sin x + B \cos x. \tag{80}$$

Setting $y(0) = 0$ we have

$$y_1(x) = \sin x. \tag{81}$$

Then setting $y(1) = 0$ we have

$$y_2(x) = \cos 1 \, \sin x - \sin 1 \cos x. \tag{82}$$

As $p(x) = 1$, we compute

$$C = p(\xi) \left[y_1(\xi) \, y_2'(\xi) - y_2(\xi) \, y_1'(\xi) \right] = \sin 1. \tag{83}$$

Thus the Green's function is

$$G(x; \xi) = \begin{cases} \dfrac{y_2(\xi) \, y_1(x)}{C} = \dfrac{\cos 1 \sin \xi - \sin 1 \cos \xi}{\sin 1} \sin x & x < \xi \\ \dfrac{y_1(\xi) \, y_2(x)}{C} = \dfrac{\sin \xi}{\sin 1} (\cos 1 \, \sin x - \sin 1 \cos x) & x > \xi \end{cases}. \tag{84}$$

The solution is then

$$\begin{aligned} y(x) &= \int_0^1 G(x; \xi) \, \mathrm{d}\xi \\ &= \int_0^x \frac{\cos 1 \, \sin x - \sin 1 \cos x}{\sin 1} \sin \xi \, \mathrm{d}\xi \\ &\quad + \int_x^1 \frac{\sin x}{\sin 1} (\cos 1 \sin \xi - \sin 1 \cos \xi) \, \mathrm{d}\xi \\ &= \frac{\cos 1 \, \sin x - \sin 1 \cos x}{\sin 1} (1 - \cos x) \\ &\quad + \frac{\sin x}{\sin 1} \big(- \cos^2 1 + \cos 1 \cos x - \sin^2 1 + \sin 1 \sin x \big) \\ &= \frac{1}{\sin 1} \Big[\cos 1 \sin x - \sin 1 \cos x - \cos 1 \sin x \cos x + \sin 1 \cos^2 x \\ &\qquad - \sin x + \sin x \cos 1 \cos x + \sin 1 \sin^2 x \Big] \\ &= \frac{1}{\sin 1} [\cos 1 \sin x - \sin 1 \cos x - \sin x + \sin 1] \\ &= 1 - \cos x + \frac{\sin x}{\sin 1} (\cos 1 - 1). \end{aligned} \tag{85}$$

3. Green's function for PDEs.

We illustrate Green's function theory for PDEs by considering the Poisson equation

$$u_{xx} + u_{yy} = f(x, y) \tag{86}$$

with various boundary conditions. The idea is to first find a particular function $\Gamma(x, y; \xi, \eta)$ such that

$$\Gamma_{xx} + \Gamma_{yy} = \delta(x - \xi, y - \eta) \tag{87}$$

and then construct Green's functions using Γ. This function Γ is called the *fundamental solution* of the operator $\partial_{xx} + \partial_{yy}$.

Remark 6. As the boundary conditions do not play a role in determining Γ, what we will carry out is a "divide-and-conquer" type strategy, dealing with the equation and the boundary conditions separately.

Such a strategy is necessary. Because solving the Green's function directly from the equation together with the boundary conditions is as hard as solving the boundary value problem itself. As we will see soon, one can cleverly take advantage of the symmetry of the domain to obtain G from Γ.

3.1. Fundamental solutions.

We need to find out a function $\Gamma(x, y; \xi, \eta)$ which satisfies

$$\Gamma_{xx} + \Gamma_{yy} = \delta(x - \xi, y - \eta). \tag{88}$$

Noticing the translation invariance of the differential operator, we can assume Γ depending only on $x-\xi$ and $y-\eta$, which reduces the problem to finding $\Gamma(x,y)$ such that

$$\Gamma_{xx}+\Gamma_{yy}=\delta(x,y). \tag{89}$$

Changing variables to polar coordinates, we have

$$\Gamma_{rr}+\frac{1}{r}\Gamma_r+\frac{1}{r^2}\Gamma_{\theta\theta}=\frac{1}{2\pi}\frac{\delta(r)}{r}. \tag{90}$$

Using the fact that both the operator $\partial_{xx}+\partial_{yy}$ and the RHS $\delta(x,\,y)$ are independent of θ (invariant under rotations, to be more precise), we assume Γ is independent of θ. Thus the equation becomes

$$\Gamma''+\frac{1}{r}\Gamma'=\frac{1}{2\pi}\frac{\delta(r)}{r}\implies r\,\Gamma''+\Gamma'=\frac{1}{2\pi}\delta(r). \tag{91}$$

This gives

$$(r\,\Gamma')'=\frac{1}{2\pi}\delta(r)\implies r\,\Gamma'=\frac{1}{2\pi}\text{ for }r>0. \tag{92}$$

Integrating one more time we have

$$\Gamma=\frac{1}{2\pi}\ln r+C. \tag{93}$$

We choose $C=0$ to obtain

$$\Gamma(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right). \tag{94}$$

3.2. The method of images.

Now we use the fundamental solution to construct Green's functions for various boundary conditions. We restrict ourselves to the Laplace operator. For a domain D with boundary B, there are two typical types of problems:

i. Dirichlet problem

$$u_{xx}+u_{yy}=f(x,y)\text{ in }D;\qquad u=g\text{ along }B; \tag{95}$$

ii. Neumann problem

$$u_{xx}+u_{yy}=f(x,y)\text{ in }D;\qquad \frac{\partial u}{\partial n}=g\text{ along }B. \tag{96}$$

A Green's function produces u from f and g.

Now intuitively we would like to construct the Green's function from the fundamental solution Γ which satisfies

$$\Gamma_{xx}+\Gamma_{yy}=\delta(x-\xi,y-\eta) \tag{97}$$

or equivalently

$$\iint(\Gamma_{xx}+\Gamma_{yy})\,f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta=f(x,y) \tag{98}$$

for any f. Let $u(x,\,y)$ be the solution to the problem, we first try to see whether Γ can serve as the Green's function. We compute

$$\begin{aligned}
u(x,y) &= \iint(\Gamma_{xx}+\Gamma_{yy})\,u(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \iint_D(\Gamma_{xx}+\Gamma_{yy})\,u(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \iint_D(\Gamma_{\xi\xi}+\Gamma_{\eta\eta})\,u(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \iint_D(\Gamma_\xi\,u)_\xi+(\Gamma_\eta\,u)_\eta\,\mathrm{d}\xi\,\mathrm{d}\eta-\iint_D\Gamma_\xi\,u_\xi+\Gamma_\eta\,u_\eta\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \int_B\frac{\partial\Gamma}{\partial n}u\,\mathrm{d}s-\iint_D(\Gamma\,u_\xi)_\xi+(\Gamma\,u_\eta)_\eta\,\mathrm{d}\xi\,\mathrm{d}\eta+\iint_D\Gamma\,(u_{\xi\xi}+u_{\eta\eta})\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \int_B\frac{\partial\Gamma}{\partial n}u\,\mathrm{d}s-\int_B\Gamma\,\frac{\partial u}{\partial n}\mathrm{d}s+\iint_D\Gamma\,(x-\xi,y-\eta)\,f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned} \tag{99}$$

The above becomes

$$u(x,y) = \int_B \frac{\partial \Gamma}{\partial n} g \, \mathrm{d}s - \int_B \Gamma \frac{\partial u}{\partial n} \mathrm{d}s + \int\int_D \Gamma(x-\xi, y-\eta) f(\xi,\eta) \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{100}$$

when we are solving a Dirichlet problem and

$$u(x,y) = \int_B \frac{\partial \Gamma}{\partial n} u \, \mathrm{d}s - \int_B \Gamma g \, \mathrm{d}s + \int\int_D \Gamma(x-\xi, y-\eta) f(\xi,\eta) \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{101}$$

when we are solving a Neumann problem.

Therefore Γ almost does the job except that in each formula there is one term (the red term) that is not known. To fix this, we make the following crucial observation:

> Let $\tilde{\Gamma}(x,y;\xi,\eta)$ be such that $R \equiv \tilde{\Gamma} - \Gamma$ satisfies
>
> $$R_{xx} + R_{yy} = R_{\xi\xi} + R_{\eta\eta} = 0 \tag{102}$$
>
> for all points (x, y) inside D and (ξ, η) inside D. Then the above formulas still hold with Γ replaced by $\tilde{\Gamma}$.

In light of this observation, it is clear that our Green's function for the Dirichlet problem is the particular $\tilde{\Gamma}$ such that $\tilde{\Gamma}(x,\, y;\, \xi,\, \eta) = 0$ for all $(\xi,\, \eta)$ on the boundary, while our Green's function for the Neumann problem is the particular $\tilde{\Gamma}$ such that $\frac{\partial \tilde{\Gamma}}{\partial n}(x,\, y;\, \xi,\, \eta) = 0$ for all $(\xi,\, \eta)$ on the boundary. Now the question becomes, how to find these particular $\tilde{\Gamma}$'s? Or equivalently, how to find the difference R? One method is to cleverly take advantage of the symmetries of the domain.

Example 7. Consider the following problem on the upper half plane

$$u_{xx} + u_{yy} = f(x,y) \text{ for } y > 0; \qquad u_y(x,0) = 0. \tag{103}$$

Find its Green's function.

Solution. We need one function $R(x,y;\xi,\eta)$ such that

$$R_{xx} + R_{yy} = R_{\xi\xi} + R_{\eta\eta} = 0 \tag{104}$$

whenever $y > 0$ and $\eta > 0$, and furthermore

$$(\Gamma + R)_y = \frac{\partial}{\partial n}[\Gamma + R] = 0 \tag{105}$$

whenever $y > 0$ and $\eta = 0$. The idea is to look for R of the form $a\,\Gamma(b\,(x' - \xi), b\,(y' - \eta))$ and try to find out appropriate a, b, x', y'.

Now as $\Gamma_{xx} + \Gamma_{yy} = \Gamma_{\xi\xi} + \Gamma_{\eta\eta} = 0$ whenever $(x,y) \neq (\xi,\eta)$, we see that the first condition on R is satisfied as long as

$$(x,y) \neq (\xi,\eta) \text{ whenever } y > 0, \eta > 0. \tag{106}$$

It suffices to take $y' < 0$.

Furthermore, to make $(\Gamma + R)_y = 0$, it suffices to make the function $\Gamma + R$ an even function in the y variable.

Combining these two observations, we see that the appropriate parameters are

$$a = 1, \quad b = 1, \quad x' = x, \quad y' = -y. \tag{107}$$

Thus the Greens function is

$$G(x,y;\xi,\eta) = \Gamma(x-\xi, y-\eta) + \Gamma(x-\xi, -y-\eta) = \frac{1}{2\pi} \ln\left(\sqrt{(x-\xi)^2 + (y-\eta)^2}\,\sqrt{(x-\xi)^2 + (y+\eta)^2}\right). \tag{108}$$