Week 04 - 05: Wave Equations

In this chapter, we will consider the (1D) wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{1}$$

As we have seen, the general solution is

$$u(x,t) = f(x + c t) + g(x - c t) \tag{2}$$

with two artibrary functions f and g. In practice, the wave equation describes (among other phenomena) the vibration of strings or membranes or propagation of sound waves. In these phenomena obviously there is no arbitrary functions involved.

Now how do we "fix" the arbitrariness? Remember that for first order PDE (with two variables), the solutions are surfaces in the space. One way to choose one from the infinitely many surfaces is the prescribe one curve in the space and require the solution surface to pass through it. Remember that the coordinates are $(x,\ y,\ u)$, thus a curve in the space is represented by values of u assigned to a curve $(x(t),\ y(t))$. Such a problem is called Cauchy problem.

When we deal with second order equations, it turns out that assigning u along a curve is not enough, we also need to assign derivatives. To see why, we can rewrite a second order eqution into a system of first order equations by assigning new variables

$$v = u_x, \quad w = u_y. \tag{3}$$

Then intuitively to "fix" one solution, we need to assign v, w along the curve too.

In this chapter, we will consider the Cauchy problem for (mostly 1D) wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{4}$$

1. D'Alembert's formula.

As a start, we study the simplest one, with u and u_t assigned along the x-axis:[1]

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad x \in \mathbb{R},\ t > 0 & (5)\\ u(x,0) &= f(x), \qquad x \in \mathbb{R} & (6)\\ u_t(x,0) &= g(x), \qquad x \in \mathbb{R}. & (7) \end{aligned}$$

From past lectures we know that the general solution is

$$\phi(x + c t) + \psi(x - c t). \tag{8}$$

We would use the values of u, u_t along the x-axis to fix the two arbitrary functions ϕ and ψ. We have

$$\begin{aligned} \phi(x) + \psi(x) = u(x,0) &= f(x), & (9)\\ c\,\phi'(x) - c\,\psi'(x) = u_t(x,0) &= g(x). & (10) \end{aligned}$$

From this we have

$$\begin{aligned} \phi'(x) + \psi'(x) &= f'(x) & (11)\\ \phi'(x) - \psi'(x) &= \frac{g(x)}{c}, & (12) \end{aligned}$$

whose solutions are

$$\begin{aligned} \phi'(x) &= \frac{1}{2}\left[f'(x) + \frac{g(x)}{c}\right], & (13)\\ \psi'(x) &= \frac{1}{2}\left[f'(x) - \frac{g(x)}{c}\right]. & (14) \end{aligned}$$

1. Think: why is assigning u_t enough?

These give

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int g(x) + C_1 \tag{15}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int g(x) + C_2 \tag{16}$$

Using

$$\phi(x) + \psi(x) = f(x) \tag{17}$$

again, we can conclude $C_1 + C_2 = 0$, that is

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y) + C \tag{18}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y) - C \tag{19}$$

As a consequence the solution formula is

$$u(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y. \tag{20}$$

This is called the *d'Alembert formula*.

Example 1. (§5.12, 1 b)) Determine the solution.

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \sin x, \quad u_t(x,0) = x^2. \tag{21}$$

Solution. We use the d'Alembert formula:

$$u(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} y^2\,\mathrm{d}y. \tag{22}$$

Some calculation yields

$$u(x,t) = \sin x \cos ct + x^2 t + \frac{1}{3}c^2 t^3. \tag{23}$$

Example 2. (§5.12, 1 e)) Determine the solution.

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \log\left(1+x^2\right), \quad u_t(x,0) = 2. \tag{24}$$

Solution. We use the d'Alembert formula:

$$u(x,t) = \frac{1}{2}\left[\log\left(1+(x+ct)^2\right) + \log\left(1+(x-ct)^2\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} 2\,\mathrm{d}y. \tag{25}$$

Some calculation gives

$$u(x,t) = \frac{1}{2}\left[\log\left(1+(x+ct)^2\right) + \log\left(1+(x-ct)^2\right)\right] + 2t. \tag{26}$$

(Note: It is not necessary to further "simplify" as the book did.

From the d'Alembert formula we see that

- the solution u at (x,t) only depends on the values of f at two points $x \pm ct$ and the values of g between these two points. In other words, what happens outside the interval $(x-ct, x+ct)$ does not affect the value $u(x,t)$ at all. If we do not treat $t=0$ specially, we see that the value $u(x,t)$ is only affected by what happens inside the cone formed by the two rays from (x,t) passing $(x-ct, 0)$ and $(x+ct, 0)$ respectively. This cone is called *domain of dependence*.
- on the other hand, if we represent $u(x,t)$ using values along $t = t_0$, we have

$$u(x,t) = u(x+c(t-t_0), t_0) + u(x-c(t-t_0), t_0) + \frac{1}{2c}\int_{x-c(t-t_0)}^{x+c(t-t_0)} u_t(y,t_0)\,\mathrm{d}y. \tag{27}$$

We see that the values $u(x_0, t_0)$ and $u_t(x_0, t_0)$ are only involved in formulas for $u(x, t)$ with $x \in [x_0 - c\,(t - t_0), x_0 + c\,(t - t_0)]$. Or equivalently, what happens at (x_0, t_0) only affects what happens in the cone formed by the two rays from (x_0, t_0) passing through $(x_0 - c\,(t - t_0), t)$ and $(x_0 + c\,(t - t_0), t)$. This cone is called the *range of influence*.

Finally we look at a general hyperbolic Cauchy problem.

Example 3. (§5.12, 4) Solve

$$u_{xx} + 2\,u_{xy} - 3\,u_{yy} = 0, \quad u(x,0) = \sin x, \quad u_y(x,0) = x. \tag{28}$$

Solution. First we need to reduce the equation to canonical form.

The characteristics equation is

$$(\mathrm{d}y)^2 - 2\,(\mathrm{d}x)\,(\mathrm{d}y) - 3\,(\mathrm{d}x)^2 = 0 \implies (\mathrm{d}y - 3\,\mathrm{d}x)\,(\mathrm{d}y + \mathrm{d}x) = 0 \tag{29}$$

which gives

$$\mathrm{d}(y - 3\,x) = 0, \qquad \mathrm{d}(y + x) = 0 \tag{30}$$

and leads to

$$\xi = y - 3\,x, \qquad \eta = y + x. \tag{31}$$

Now we perform the change of variables. Calculate

$$\xi_x = -3, \;\; \xi_y = 1, \;\; \xi_{xx} = \xi_{xy} = \xi_{yy} = 0, \tag{32}$$

$$\eta_x = 1, \;\; \eta_y = 1, \;\; \eta_{xx} = \eta_{xy} = \eta_{yy} = 0. \tag{33}$$

This gives

$$u_{xx} = 9\,u_{\xi\xi} - 6\,u_{\xi\eta} + u_{\eta\eta}, \tag{34}$$
$$u_{xy} = -3\,u_{\xi\xi} - 2\,u_{\xi\eta} + u_{\eta\eta}, \tag{35}$$
$$u_{yy} = u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta}, \tag{36}$$

This transforms the equation to

$$0 = (9\,u_{\xi\xi} - 6\,u_{\xi\eta} + u_{\eta\eta}) + 2\,(-3\,u_{\xi\xi} - 2\,u_{\xi\eta} + u_{\eta\eta}) - 3\,(u_{\xi\xi} + 2\,u_{\xi\eta} + u_{\eta\eta}) = -16\,u_{\xi\eta}. \tag{37}$$

Thus the general solution is

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta) \implies u(x, y) = \phi(y - 3\,x) + \psi(y + x). \tag{38}$$

Now the Cauchy data leads to

$$\phi(-3\,x) + \psi(x) = u(x, 0) = \sin x, \tag{39}$$

$$\phi'(-3\,x) + \psi'(x) = u_y(x, 0) = x. \tag{40}$$

Taking derivative of the first equation we have

$$-3\,\phi'(-3\,x) + \psi'(x) = \cos x. \tag{41}$$

Thus we can solve

$$\phi'(-3\,x) = \frac{x - \cos x}{4} \Longrightarrow \phi'(x) = \frac{-x/3 - \cos(-x/3)}{4}, \quad \psi'(x) = \frac{\cos x + 3\,x}{4}. \tag{42}$$

Integrate, we obtain

$$\phi(x) = -\frac{x^2}{24} + \frac{3}{4}\sin(-x/3) + C_1, \quad \psi(x) = \frac{3\,x^2}{8} + \frac{1}{4}\sin x + C_2. \tag{43}$$

The Cauchy data

$$\phi(-3\,x) + \psi(x) = u(x, 0) = \sin x \tag{44}$$

then gives $C_1+C_2=0$. Finally

$$u(x,y)=\phi(y-3x)+\psi(y+x)=-\frac{(y-3x)^2}{24}+\frac{3(y+x)^2}{8}+\frac{1}{4}\sin(y+x)+\frac{3}{4}\sin(x-y/3). \tag{45}$$

This can be further simplified to

$$u(x,y)=\frac{y^2}{3}+xy+\frac{1}{4}\sin(y+x)+\frac{3}{4}\sin(x-y/3). \tag{46}$$

2. Initial-boundary value problems.

2.1. Semi-infinite string with a fixed end.

We consider the case

$$\begin{aligned}
u_{tt}-c^2u_{xx} &= 0, \qquad 0<x<\infty, \quad t>0 && (47)\\
u(x,0) &= f(x), \qquad 0\leqslant x<\infty, && (48)\\
u_t(x,0) &= g(x), \qquad 0\leqslant x<\infty, && (49)\\
u(0,t) &= 0, \qquad 0\leqslant t<\infty. && (50)
\end{aligned}$$

Remark 4. Obviously to obtain continuous solutions, we need to require the consistency condition $f(0)=g(0)=0$ and $f''(x)=0$.

Again we try to fix the two arbitrary functions in the formula

$$\phi(x+ct)+\psi(x-ct). \tag{51}$$

Using the initial-boundary values we have

$$\begin{aligned}
\phi(x)+\psi(x) &= f(x), \qquad 0\leqslant x<\infty && (52)\\
\phi'(x)-\psi'(x) &= \frac{g(x)}{c}, \qquad 0\leqslant x<\infty && (53)\\
\phi(ct)+\psi(-ct) &= 0, \qquad 0\leqslant t<\infty. && (54)
\end{aligned}$$

From the first two equations we obtain

$$\begin{aligned}
\phi'(x) &= \frac{1}{2}\left[f'(x)+\frac{g(x)}{c}\right], \qquad 0\leqslant x<\infty && (55)\\
\psi'(x) &= \frac{1}{2}\left[f'(x)-\frac{g(x)}{c}\right], \qquad 0\leqslant x<\infty && (56)
\end{aligned}$$

which gives us

$$\phi(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_0^x g(y)+C, \quad \psi(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_0^x g(y)-C \tag{57}$$

but only for $x\geqslant 0$.

To determine ϕ and ψ for $x<0$, we try to use the boundary value along the t-axis. We have

$$\psi(x)=-\phi(-x)=-\frac{1}{2}f(-x)-\frac{1}{2c}\int_0^{-x} g(y)-C, \qquad x<0. \tag{58}$$

But what about $\phi(x)$ for $x<0$? It turns out that since we only need $u(x,t)$ for $x>0$, $t>0$, $x+ct$ is always positive, as a consequence we already have all $\phi(x+ct)$ we need. The formula would be

$$u(x,t)=\begin{cases}\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(y)\,\mathrm{d}y & x>ct\\ \frac{1}{2}[f(x+ct)-f(ct-x)]+\frac{1}{2c}\int_{ct-x}^{x+ct}g(y)\,\mathrm{d}y & 0\leqslant x<ct\end{cases}. \tag{59}$$

Example 5. (§5.12, 7) Determine the solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} &= 4\,u_{xx}, & 0<x<\infty,\ t>0 && (60)\\ u(x,0) &= x^4, & 0\leqslant x<\infty, && (61)\\ u_t(x,0) &= 0, & 0\leqslant x<\infty, && (62)\\ u(0,t) &= 0, & t\geqslant 0. && (63)\end{aligned}$$

Solution. We apply the formula. First in this problem $c=2,\ f(x)=x^4,\ g(x)=0$.

When $x>2\,t$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+2\,t)+f(x-2\,t)\right]+\frac{1}{4}\int_{x-2t}^{x+2t} g(y)\,\mathrm{d}y\\ &= \frac{1}{2}\left[(x+2\,t)^4+(x-2\,t)^4\right]+\frac{1}{4}\int_{x-2t}^{x+2t} 0\,\mathrm{d}y\\ &= x^4+24\,x^2\,t^2+16\,t^4. \end{aligned} \tag{64}$$

When $x<2\,t$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+2\,t)-f(2\,t-x)\right]+\frac{1}{4}\int_{2t-x}^{x+2t} g(y)\,\mathrm{d}y\\ &= \frac{1}{2}\left[(x+2\,t)^4-(2\,t-x)^4\right]+\frac{1}{4}\int_{2t-x}^{x+2t} 0\,\mathrm{d}y\\ &= \frac{1}{2}\Big[\left(x^4+8\,x^3\,t+24\,x^2\,t^2+32\,x\,t^3+16\,t^4\right)\\ &\quad -\left(x^4-8\,x^3\,t+24\,x^2\,t^2-32\,x\,t^3+16\,t^4\right)\Big]\\ &= 8\,x^3\,t+32\,x\,t^3. \end{aligned} \tag{65}$$

Example 6. (§5.12, 10) Consider the initial-boundary value problem

$$\begin{aligned} u_{tt} &= c^2\,u_{xx}, & 0<x<\infty, \quad t>0, && (66)\\ u(x,0) &= f(x), & 0\leqslant x<\infty, && (67)\\ u_t(x,0) &= g(x), & 0\leqslant x<\infty, && (68)\\ u(0,t) &= 0, & t\geqslant 0. && (69)\end{aligned}$$

(Note that there are several typos in the book). Show that one can solve it by extending f,g oddly.
Solution. We extend f and g:

$$\tilde{f}(x)=\begin{cases} f(x) & x\geqslant 0\\ -f(-x) & x\leqslant 0\end{cases}, \qquad \tilde{g}(x)=\begin{cases} g(x) & x\geqslant 0\\ -g(-x) & x\leqslant 0\end{cases}. \tag{70}$$

and solve the initial value problem

$$\begin{aligned} \tilde{u}_{tt} &= c^2\,\tilde{u}_{xx}, & -\infty<x<\infty, \quad t>0, && (71)\\ \tilde{u}(x,0) &= \tilde{f}(x), & -\infty<x<\infty, && (72)\\ \tilde{u}_t(x,0) &= \tilde{g}(x), & -\infty<x<\infty. && (73)\end{aligned}$$

The d'Alembert formula gives

$$\tilde{u}(x,t)=\frac{1}{2}\left[\tilde{f}(x+c\,t)+\tilde{f}(x-c\,t)\right]+\frac{1}{2\,c}\int_{x-ct}^{x+ct}\tilde{g}(y)\,\mathrm{d}y. \tag{74}$$

From this we clearly have for $x\geqslant 0, t\geqslant 0$ (note that $x+c\,t$ is always $\geqslant 0$)

$$u(x,t)=\begin{cases} \frac{1}{2}\left[f(x+c\,t)+f(x-c\,t)\right]+\frac{1}{2\,c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y & x>c\,t\\ \frac{1}{2}\left[f(x+c\,t)-f(c\,t-x)\right]+\frac{1}{2\,c}\int_{ct-x}^{x+ct} g(y)\,\mathrm{d}y & 0\leqslant x<c\,t\end{cases}. \tag{75}$$

2.2. Semi-infinite string with a free end.

The problem is

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < \infty, \quad t > 0 && (76)\\
u(x,0) &= f(x), \quad 0 \leqslant x < \infty, && (77)\\
u_t(x,0) &= g(x), \quad 0 \leqslant x < \infty, && (78)\\
u_x(0,t) &= 0, \qquad 0 \leqslant t < \infty. && (79)
\end{aligned}$$

Similar to the above case, we first write

$$u(x,t) = \phi(x + c\,t) + \psi(x - c\,t) \tag{80}$$

Now using the initial value, we obtain

$$\begin{aligned}
\phi(x) + \psi(x) &= f(x), \quad x \geqslant 0 && (81)\\
\phi'(x) - \psi'(x) &= \frac{g(x)}{c}, \quad x \geqslant 0 && (82)
\end{aligned}$$

which gives

$$\phi(x) = \frac{1}{2}\,f(x) + \frac{1}{2\,c}\int_0^x g(y) + C, \quad \psi(x) = \frac{1}{2}\,f(x) - \frac{1}{2\,c}\int_0^x g(y) - C \tag{83}$$

For $\psi(x)$ with $x < 0$, we use

$$\phi'(c\,t) + \psi'(-c\,t) = 0 \Longrightarrow \psi'(x) = -\phi'(-x) \text{ for } x < 0. \tag{84}$$

This leads to

$$\psi'(x) = -\frac{1}{2}f'(-x) - \frac{1}{2\,c}\,g(-x) \Longrightarrow \psi(x) = \frac{1}{2}\,f(-x) + \frac{1}{2\,c}\int^{-x} g(y) + C' \qquad x < 0 \tag{85}$$

The constant C' is determined by requiring $\psi(x)$ to be continuous, that is the two formulas should yield the same value at $x = 0$, therefore $C' = -C$.

Putting everything together we have

$$u(x,t) = \begin{cases} \dfrac{1}{2}\,[f(x + c\,t) + f(x - c\,t)] + \dfrac{1}{2\,c}\displaystyle\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y & x > c\,t \\ \dfrac{1}{2}\,[f(x + c\,t) + f(c\,t - x)] + \dfrac{1}{2\,c}\left[\displaystyle\int_0^{x+ct} g(y)\,\mathrm{d}y + \int_0^{ct-x} g(y)\,\mathrm{d}y\right] & 0 \leqslant x < c\,t \end{cases}. \tag{86}$$

Example 7. (§5.12, 8) Determine the solution of the initial-boundary value problem

$$\begin{aligned}
u_{tt} &= 9\,u_{xx}, \qquad 0 < x < \infty, \; t > 0 && (87)\\
u(x,0) &= 0, \qquad 0 \leqslant x < \infty && (88)\\
u_t(x,0) &= x^3, \qquad 0 \leqslant x < \infty && (89)\\
u_x(0,t) &= 0. \qquad 0 \leqslant t < \infty && (90)
\end{aligned}$$

Solution. Here $c = 3,\ f = 0,\ g = x^3$. Using the formula we have

when $x > 3\,t$,

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\,[f(x + 3\,t) + f(x - 3\,t)] + \frac{1}{6}\int_{x-3t}^{x+3t} g(y)\,\mathrm{d}y \\
&= \frac{1}{6}\int_{x-3t}^{x+3t} y^3\,\mathrm{d}y \\
&= \frac{1}{24}\left[(x + 3\,t)^4 - (x - 3\,t)^4\right] \\
&= \frac{1}{24}\left[24\,x^3\,t + 216\,x\,t^3\right] \\
&= x^3\,t + 9\,x\,t^3. && (91)
\end{aligned}$$

when $x < 3\,t$,

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x+3\,t)+f(3\,t-x)]+\frac{1}{6}\left[\int_0^{x+3t} g(y)\,\mathrm{d}y+\int_0^{3t-x} g(y)\,\mathrm{d}y\right] \\ &= \frac{1}{6}\left[\int_0^{x+3t} y^3\,\mathrm{d}y+\int_0^{3t-x} y^3\,\mathrm{d}y\right] \\ &= \frac{1}{24}\left[(x+3\,t)^4+(3\,t-x)^4\right] \\ &= \frac{1}{24}\left[2\,x^4+108\,x^2\,t^2+162\,t^4\right] \\ &= \frac{1}{12}\,x^4+\frac{9}{2}\,x^2\,t^2+\frac{27}{4}\,t^4. \end{aligned} \tag{92}$$

2.3. Equations with nonhomogeneous boundary conditions.

Consider

$$u_{tt}-c^2\,u_{xx} = 0, \qquad 0<x<\infty, \quad t>0 \tag{93}$$
$$u(x,0) = f(x), \quad 0\leqslant x<\infty, \tag{94}$$
$$u_t(x,0) = g(x), \quad 0\leqslant x<\infty, \tag{95}$$
$$u(0,t) = p(t), \qquad 0\leqslant t<\infty. \tag{96}$$

Remark 8. The consistency conditions are

$$p(0)=f(0), \quad p'(0)=g(0), \quad p''(0)=c^2\,f''(0). \tag{97}$$

The approach is similar to that in the homogeneous case, for $x<0$ we obtain

$$\psi(x)=p\Big(-\frac{x}{c}\Big)-\phi(-x). \tag{98}$$

As a consequence, the formula becomes

$$u(x,t)=\begin{cases} \frac{1}{2}[f(x+c\,t)+f(x-c\,t)]+\frac{1}{2\,c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y & x>c\,t \\ \frac{1}{2}[f(x+c\,t)-f(c\,t-x)]+\frac{1}{2\,c}\int_{ct-x}^{x+ct} g(y)\,\mathrm{d}y+p\Big(t-\frac{x}{c}\Big) & 0\leqslant x<c\,t \end{cases}. \tag{99}$$

Our last case is

$$u_{tt}-c^2\,u_{xx} = 0, \qquad 0<x<\infty, \quad t>0 \tag{100}$$
$$u(x,0) = f(x), \quad 0\leqslant x<\infty, \tag{101}$$
$$u_t(x,0) = g(x), \quad 0\leqslant x<\infty, \tag{102}$$
$$u_x(0,t) = q(t), \qquad 0\leqslant t<\infty. \tag{103}$$

We obtain

$$\psi(x)=\phi(-x)-c\int_0^{-x/c} q(y)\,\mathrm{d}y-C. \tag{104}$$

and the solution for $x<c\,t$ is given by

$$\frac{1}{2}[f(x+c\,t)+f(c\,t-x)]+\frac{1}{2\,c}\left[\int_0^{x+ct} g(y)\,\mathrm{d}y+\int_0^{ct-x} g(y)\,\mathrm{d}y\right]-c\int_0^{t-x/c} q(y)\,\mathrm{d}y. \tag{105}$$

3. Vibration of finite string with fixed ends.

A more complicated problem is when we have two boundaries.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < l, \quad t > 0 && (106)\\ u(x,0) &= f(x), \quad 0 \leqslant x < l, && (107)\\ u_t(x,0) &= g(x), \quad 0 \leqslant x < l, && (108)\\ u(0,t) &= 0, \qquad 0 \leqslant t < \infty && (109)\\ u(l,t) &= 0, \qquad 0 \leqslant t < \infty. && (110)\end{aligned}$$

Once more we start with the general solution

$$u(x,t) = \phi(x + c\,t) + \psi(x - c\,t) \tag{111}$$

and try to exclude arbitrariness using the initial and boundary values.

This time, from the initial values we can only obtain

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2\,c} \int_0^x g(y)\,\mathrm{d}y + C, \quad \psi(x) = \frac{1}{2} f(x) - \frac{1}{2\,c} \int_0^x g(y)\,\mathrm{d}y - C, \quad 0 \leqslant x \leqslant l. \tag{112}$$

But this only gives us the solution inside

$$\{(x,t) \colon x + c\,t, x - c\,t \in [0,l]\}. \tag{113}$$

What to do for points outside? We need to use the boundary conditions. Using them we obtain

$$\psi(x) = -\phi(-x), \qquad \phi(x) = -\psi(2\,l - x) \tag{114}$$

which extends the definition of ψ to $[-l, l]$ and that of ϕ to $[0, 2\,l]$. Now using these two relations again, we obtain ψ over $[-2\,l, l]$ while ϕ over $[0, 3\,l]$. Repeating this process, we finally got ψ defined over $(-\infty, l]$ while ϕ over $[0, \infty)$. It is clear that this is sufficient for determine $u(x,t)$ using

$$u(x,t) = \phi(x + c\,t) + \psi(x - c\,t) \tag{115}$$

for all $(x,t) \colon 0 \leqslant x \leqslant l, t \geqslant 0$.

Example 9. (§5.12, 15) Find the solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} &= 4\,u_{xx}, \qquad 0 < x < 1,\ t > 0 && (116)\\ u(x,0) &= 0, \qquad 0 \leqslant x \leqslant 1, && (117)\\ u_t(x,0) &= x\,(1-x), \qquad 0 \leqslant x \leqslant 1, && (118)\\ u(0,t) &= 0, \qquad t \geqslant 0, && (119)\\ u(1,t) &= 0. \qquad t \geqslant 0. && (120)\end{aligned}$$

Solution. In this problem $c = 2$, $f = 0$, $g = x\,(1-x)$. Recall that the general solution is

$$u(x,t) = \phi(x + 2\,t) + \psi(x - 2\,t). \tag{121}$$

As we need $u(x,t)$ for all $0 < x < 1$, $t > 0$, we need values of $\phi(x)$ for all $x > 0$ and $\psi(x)$ for all $x < 1$.

First using the initial values, we conclude

$$\begin{aligned} \phi(x) &= \frac{1}{4} \int^x y\,(1-y)\,\mathrm{d}y + C = \frac{1}{8} x^2 - \frac{1}{12} x^3 + C && (122)\\ \psi(x) &= -\frac{1}{8} x^2 + \frac{1}{12} x^3 - C. && (123)\end{aligned}$$

Both formulas hold only for $0 \leqslant x \leqslant 1$.

Now using $u(0,t) = 0$ we obtain

$$\phi(c\,t) + \psi(-c\,t) = 0 \Longrightarrow \psi(x) = -\phi(-x), \tag{124}$$

using $u(1,t) = 0$ we obtain

$$\phi(1 + c\,t) + \psi(1 - c\,t) = 0 \Longrightarrow \phi(x) = -\psi(2 - x). \tag{125}$$

Thus we will use $\psi(x) = -\phi(-x)$ and $\phi(x) = \psi(2 - x)$ to determine values of ϕ and ψ outside $0 \leqslant x \leqslant 1$.

Using $\psi(x) = -\phi(-x)$ once, we obtain

$$\psi(x) = -\frac{1}{8}x^2 - \frac{1}{12}x^3 - C \qquad -1 \leqslant x \leqslant 0. \tag{126}$$

Next using $\phi(x) = -\psi(2-x)$ we have

$$\phi(x) = \frac{1}{8}(2-x)^2 - \frac{1}{12}(2-x)^3 + C \qquad 1 \leqslant x \leqslant 2, \tag{127}$$

and

$$\phi(x) = \frac{1}{8}(2-x)^2 + \frac{1}{12}(2-x)^3 + C = \frac{1}{8}(x-2)^2 - \frac{1}{12}(x-2)^3 + C \qquad 2 \leqslant x \leqslant 3. \tag{128}$$

Note that so far we have determined ψ on $-1 \leqslant x \leqslant 1$ and ϕ on $0 \leqslant x \leqslant 3$.

Now using $\psi(x) = -\phi(-x)$ again, we can determine ψ on $-3 \leqslant x \leqslant -1$:

$$\psi(x) = -\frac{1}{8}(2+x)^2 + \frac{1}{12}(2+x)^3 - C \qquad -2 \leqslant x \leqslant -1, \tag{129}$$

$$\psi(x) = -\frac{1}{8}(2+x) + \frac{1}{12}(-x-2)^3 - C = -\frac{1}{8}(2+x) - \frac{1}{12}(2+x)^3 - C. \tag{130}$$

Comparing the values of ψ over $-1 \leqslant x \leqslant 1$ and $-3 \leqslant x \leqslant -1$, we observe that ψ on $-3 \leqslant x \leqslant -1$ can be obtained from ψ over $-1 \leqslant x \leqslant 1$ by a translation.

Carrying on the process, we see that ϕ and ψ are in fact obtained from periodic extension of their values over $0 \leqslant x \leqslant 2$ and $-1 \leqslant x \leqslant 1$, respectively.

Remark 10. As we can see from the past few lectures, the closed solution formulas are less and less helpful in understanding the equation/solution as the domain gets more complicated. In a few weeks, we will study a different approach to these "more complicated" initial-boundary value problems.

4. Nonhomogeneous wave equations.

In this section we consider the initial value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), \qquad x \in \mathbb{R}, \quad t > 0 && (131)\\ u(x,0) &= f(x), \qquad x \in \mathbb{R} && (132)\\ u_t(x,0) &= g(x), \qquad x \in \mathbb{R}. && (133) \end{aligned}$$

Note that this time we do not have the formula for general solutions at hand. Nevertheless, we can try to simplify the equation a bit using the d'Alembert formula. Let v be defined by the d'Alembert's formula

$$v(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y. \tag{134}$$

We know that v solves

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0, \qquad x \in \mathbb{R}, \quad t > 0 && (135)\\ v(x,0) &= f(x), \qquad x \in \mathbb{R} && (136)\\ v_t(x,0) &= g(x), \qquad x \in \mathbb{R}. && (137) \end{aligned}$$

Let $w = u - v$. Using the linearity of the equation, we easily see that w solves

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= h(x,t), \qquad x \in \mathbb{R}, \quad t > 0 && (138)\\ w(x,0) &= 0, \qquad x \in \mathbb{R} && (139)\\ w_t(x,0) &= 0, \qquad x \in \mathbb{R}. && (140) \end{aligned}$$

All we need to do is to solve w.

In the following we will forget the physical meaning that t is time and x space, and treat them just as the two coordinate variables for the plane. To find out $w(x_0,\ t_0)$, we integrate the equation over the domain of dependence:

$$\Omega \equiv \{(x,t):\ t > 0,\ x \in (x_0 - c(t_0 - t), x_0 + c(t_0 - t))\}. \tag{141}$$

We have

$$\iint_\Omega \left(w_{tt} - c^2\, w_{xx}\right) \mathrm{d}x\, \mathrm{d}t = \iint_\Omega h(x,t)\, \mathrm{d}x\, \mathrm{d}t. \tag{142}$$

Now we use Green's formula

$$\iint_\Omega F_x + G_y\, \mathrm{d}x\, \mathrm{d}y = \oint_{\partial\Omega} F\, \mathrm{d}y - G\, \mathrm{d}x \tag{143}$$

to obtain (with $t \longleftrightarrow y$, $w_t \longleftrightarrow G$, $-c^2\, w_x \longleftrightarrow F$)

$$\iint_\Omega h(x,t)\, \mathrm{d}x\, \mathrm{d}t = -\oint_{\partial\Omega} c^2\, w_x\, \mathrm{d}t + w_t\, \mathrm{d}x. \tag{144}$$

Remark 11. One way to remember Green's formula is the following. Consider the special case that $\Omega = \{(x,y)\colon\ 0 < x, y < 1\}$. Then

$$\begin{aligned}
\iint_\Omega F_x + G_y\, \mathrm{d}x\, \mathrm{d}y &= \int_0^1 \left[\int_0^1 F_x\, \mathrm{d}x\right] \mathrm{d}y + \int_0^1 \left[\int_0^1 G_y\, \mathrm{d}y\right] \mathrm{d}x \\
&= \int_0^1 F(1,y)\, \mathrm{d}y - \int_0^1 F(0,y)\, \mathrm{d}y + \int_0^1 G(1,x)\, \mathrm{d}x - \int_0^1 G(0,x)\, \mathrm{d}x \\
&= \int_0^1 F(1,y)\, \mathrm{d}y + \int_1^0 F(0,y)\, \mathrm{d}y - \left[\int_0^1 G(0,x)\, \mathrm{d}x + \int_1^0 G(1,x)\, \mathrm{d}x\right] \\
&= \oint_{\partial\Omega} F\, \mathrm{d}y - G\, \mathrm{d}x.
\end{aligned} \tag{145}$$

The boundary $\partial\Omega$ can be consists of three parts: Γ_1 along the x-axis, Γ_2 along $x + c\,t = x_0 + c\,t_0$, Γ_3 along $x - c\,t = x_0 - c\,t_0$. We evaluate the integrals on them one by one.

- Γ_1. Along it we have $\mathrm{d}y = 0$ and $w_t = 0$. Thus $\int_{\Gamma_1} = 0$.
- Γ_2. We represent Γ_2 by $\{(x(t), t)\}$ with $x(t) + c\,t = x_0 + c\,t_0$. Then we have $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -c$ and

$$\begin{aligned}
\int_{\Gamma_2} c^2\, w_x\, \mathrm{d}t + w_t\, \mathrm{d}x &= \int_0^{t_0} c^2 w_x(x(t),t)\, \mathrm{d}t + \int_0^{t_0} w_t(x(t),t)\, \mathrm{d}x(t) \\
&= \int_0^{t_0} c^2 w_x(x(t),t)\, \mathrm{d}t + \int_0^{t_0} w_t(x(t),t)\, (-c)\, \mathrm{d}t \\
&= \int_0^{t_0} -c\left[w_x(x(t),t)\, \frac{\mathrm{d}x(t)}{\mathrm{d}t} + w_t(x(t),t)\right] \mathrm{d}t \\
&= -c \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}t} w(x(t),t)\, \mathrm{d}t \\
&= -c\, [w(x(t_0), t_0) - w(x(0), 0)] \\
&= -c\, w(x_0, t_0).
\end{aligned} \tag{146}$$

- Γ_3 is similar to Γ_2.

Putting the three parts together we obtain

$$\iint_\Omega h(x,t)\, \mathrm{d}x\, \mathrm{d}t = -\oint_{\partial\Omega} c^2\, w_x\, \mathrm{d}t + w_t\, \mathrm{d}x = 2\, c\, w(x_0, t_0) \tag{147}$$

which gives the formula

$$w(x,t) = \frac{1}{2\,c} \iint_\Omega h(y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{148}$$

Remark 12. An equivalent approach is to use Gauss' theorem instead of Green's formula.

$$\iint_\Omega F_t + G_x\, \mathrm{d}x\, \mathrm{d}t = \oint_{\partial\Omega} F n_t + G\, n_x\, \mathrm{d}s \tag{149}$$

where s is the arc length variable along the boundary, and n_t, n_x are the t-component and x-component of the outer normal $\boldsymbol{n}$. Taking $F = w_t$ and $G = -c^2 w_x$, we have

$$\int\int_\Omega \left(w_{tt} - c^2\, w_{xx}\right) \mathrm{d}x\, \mathrm{d}t = \oint_{\partial\Omega} w_t\, n_t - c^2\, w_x\, n_x\, \mathrm{d}s$$

The boundary $\partial\Omega$ consists of three parts. Along the two sides we have

$$\begin{pmatrix} n_t \\ n_x \end{pmatrix} = \begin{pmatrix} \pm\frac{c}{\sqrt{1+c^2}} \\ \frac{1}{\sqrt{1+c^2}} \end{pmatrix}. \tag{150}$$

As a consequence, we have

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \pm\frac{1}{\sqrt{1+c^2}}, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{c}{\sqrt{1+c^2}} \tag{151}$$

along them, which means

$$\frac{1}{\sqrt{1+c^2}}\, w_t \mp \frac{c}{\sqrt{1+c^2}}\, w_x = \frac{\mathrm{d}w}{\mathrm{d}s} \tag{152}$$

therefore we finally have

$$2\,c\,u(x_0, t_0) = \int\int_\Omega h(x,t)\, \mathrm{d}x\, \mathrm{d}t. \tag{153}$$

The general formula for non-homogeneous problem is then

$$u(x,t) = \frac{1}{2}\left[f(x+c\,t) + f(x-c\,t)\right] + \frac{1}{2\,c}\int_{x-ct}^{x+ct} g(y)\, \mathrm{d}y + \frac{1}{2\,c}\int\int_{\Omega(x,t)} h(y,s)\, \mathrm{d}y\, \mathrm{d}s \tag{154}$$

where

$$\Omega(x,t) = \{(y,s):\ s>0,\ y \in (x - c\,(t-s), x + c\,(t-s))\}. \tag{155}$$

Note. The solution formula in the book (5.7.11) is not really useful as the authors forgot to replace h by h^*. Please take a few minutes to compare (5.7.11) in the book and our formula above.

Example 13. (§5.12, 2 b)) Solve

$$u_{tt} - c^2\, u_{xx} = x + c\,t, \quad u(x,0) = x, \quad u_t(x,0) = \sin x. \tag{156}$$

Solution. Here $f(x) = x$, $g(x) = \sin x$, $h(x,t) = x + c\,t$. We evaluate

$$\int_{x-ct}^{x+ct} \sin y\, \mathrm{d}y = -\left[\cos(x+c\,t) - \cos(x-c\,t)\right] = 2\sin x\, \sin c\,t. \tag{157}$$

and

$$\begin{aligned} \int\int_{\Omega(x,t)} y + c\,s\, \mathrm{d}y\, \mathrm{d}s &= \int_0^t \left[\int_{x-c(t-s)}^{x+c(t-s)} y + c\,s\, \mathrm{d}y\right] \mathrm{d}s \\ &= \int_0^t 2\,x\,c\,(t-s) + 2\,c^2\,s\,(t-s)\, \mathrm{d}s \\ &= 2\,c\,x\,t^2 - x\,c\,t^2 + c^2\,t^3 - \frac{2}{3}\,c^2\,t^3 \\ &= c\,x\,t^2 + \frac{1}{3}\,c^2\,t^3. \end{aligned} \tag{158}$$

Thus the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+c\,t) + f(x-c\,t)\right] + \frac{1}{2\,c}\int_{x-ct}^{x+ct} g(y)\, \mathrm{d}y + \frac{1}{2\,c}\int\int_{\Omega(x,t)} h(y,s)\, \mathrm{d}y\, \mathrm{d}s \\ &= x + \frac{1}{c}\sin x\, \sin c\,t + \frac{1}{2}\,x\,t^2 + \frac{c}{6}\,t^3. \end{aligned} \tag{159}$$

Substituting back into the equation, we see that we have found the correct answer.

Example 14. (§5.12, 2 e)) Solve

$$u_{tt} - c^2\, u_{xx} = x\,e^t, \quad u(x,0) = \sin x, \quad u_t(x,0) = 0. \tag{160}$$

Solution. We have $f(x) = \sin x,\ g(x) = 0,\ h(x,t) = x\,e^t$. Calculate

$$\begin{aligned}
\iint_{\Omega(x,t)} h(y,s)\,\mathrm{d}y\,\mathrm{d}s &= \iint_{\Omega(x,t)} y\,e^s\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_0^t \left[\int_{x-c(t-s)}^{x+c(t-s)} y\,e^s\,\mathrm{d}y\right]\mathrm{d}s \\
&= \int_0^t \frac{1}{2}\left[(x+c\,(t-s))^2 - (x-c\,(t-s))^2\right] e^s\,\mathrm{d}s \\
&= \int_0^t 2\,x\,c\,(t-s)\,e^s\,\mathrm{d}s \\
&= 2\,x\,c\left[t\int_0^t e^s\,\mathrm{d}s - \int_0^t s\,e^s\,\mathrm{d}s\right] \\
&= 2\,x\,c\,t\,(e^t-1) - 2\,x\,c\,t\,e^t + 2\,x\,c\,(e^t-1) \\
&= 2\,x\,c\,(e^t - t - 1). \qquad (161)
\end{aligned}$$

where the latter integral is evaluated using integration by parts:

$$\int_0^t s\,e^s\,\mathrm{d}s = \int_0^t s\,\mathrm{d}e^s = s\,e^s\,|_0^t - \int_0^t e^s\,\mathrm{d}s = t\,e^t - [e^t - 1]. \qquad (162)$$

Putting everything together we obtain the solution

$$u(x,t) = \sin x \cos c\,t + x\,(e^t - t - 1). \qquad (163)$$

5. Spherical wave equations.

One can also consider wave equations in higher dimensions. For example, in 2D we have

$$\begin{aligned}
u_{tt} - c^2\,(u_{xx} + u_{yy}) &= 0 \qquad (164) \\
u(x,y,0) &= f(x,y) \qquad (165) \\
u_t(x,y,0) &= g(x,y), \qquad (166)
\end{aligned}$$

in 3D we have

$$\begin{aligned}
u_{tt} - c^2\,(u_{xx} + u_{yy} + u_{zz}) &= 0 \qquad (167) \\
u(x,y,z,0) &= f(x,y,z) \qquad (168) \\
u_t(x,y,z,0) &= g(x,y,z). \qquad (169)
\end{aligned}$$

Solution formulas are also available but their derivation is beyond the scope of our course here. Nevertheless, we will deal with a special case in 3D, where all functions only depends on the radius $r = \sqrt{x^2+y^2+z^2}$.

First we need to re-write the equations so that only t and r appear. Using chain rule we obtain

$$\begin{aligned}
u(r,t)_{xx} &= (u_x)_x = (u_r\,r_x)_x = \left(u_r\,\frac{x}{r}\right)_x = u_{rr}\,\frac{x^2}{r^2} + \frac{1}{r}\,u_r - \frac{x^2}{r^3}\,u_r, \qquad (170) \\
u(r,t)_{yy} &= u_{rr}\,\frac{y^2}{r^2} + \frac{1}{r}\,u_r - \frac{y^2}{r^3}\,u_r, \qquad (171) \\
u(r,t)_{zz} &= u_{rr}\,\frac{z^2}{r^2} + \frac{1}{r}\,u_r - \frac{z^2}{r^3}\,u_r. \qquad (172)
\end{aligned}$$

Using the fact that $x^2 + y^2 + z^2 = r^2$ we obtain

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r}\,u_r. \qquad (173)$$

The equations then become

$$\begin{aligned}
u_{tt} - c^2\left(u_{rr} + \frac{2}{r}\,u_r\right) &= 0 \qquad r>0, t>0 \qquad (174) \\
u(r,0) &= f(r) \qquad r \geqslant 0 \qquad (175) \\
u_t(r,0) &= g(r) \qquad r \geqslant 0 \qquad (176)
\end{aligned}$$

This equation can be explicitly solved through the following trick: Let

$$U(r,t) \equiv r\,u(r,t). \tag{177}$$

Then we have

$$\begin{aligned} U_{tt} - c^2 U_{rr} &= 0 & r>0, t>0 \quad (178)\\ U(r,0) &= F(r) \equiv r\,f(r) & r \geqslant 0 \quad (179)\\ U_t(r,0) &= G(r) \equiv r\,g(r) & r \geqslant 0 \quad (180)\end{aligned}$$

If we assume u to be bounded, then $U(0,t)=0$ for all $t \geqslant 0$. Thus the equation for U can be solved using the formula we have discussed:

$$U(r,t) = \begin{cases} \frac{1}{2}[F(r+c\,t) + F(r-c\,t)] + \frac{1}{2\,c}\int_{r-ct}^{r+ct} G(\rho)\,\mathrm{d}\rho & r > c\,t \\ \frac{1}{2}[F(r+c\,t) - F(c\,t-r)] + \frac{1}{2\,c}\int_{ct-r}^{r+ct} G(\rho)\,\mathrm{d}\rho & 0 \leqslant r \leqslant c\,t \end{cases}. \tag{181}$$

Recalling $U = r\,u$, we obtain

$$u(r,t) = \begin{cases} \frac{1}{2\,r}[(r+c\,t)\,f(r+c\,t) + (r-c\,t)\,f(r-c\,t)] + \frac{1}{2\,r\,c}\int_{r-ct}^{r+ct} \rho\,g(\rho)\,\mathrm{d}\rho & r > c\,t \\ \frac{1}{2\,r}[(r+c\,t)\,f(r+c\,t) - (c\,t-r)\,f(c\,t-r)] + \frac{1}{2\,r\,c}\int_{ct-r}^{r+ct} \rho\,g(\rho)\,\mathrm{d}\rho & 0 \leqslant r \leqslant c\,t \end{cases} \tag{182}$$